LINEAR OPERATIONS ON SUMMABLE FUNCTIONS*

BY

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In the last few years the work of Gelfand [17, 18],‡ Kantorovitch [23, 24, 25], Dunford [9, 10], Vulich [24, 25, 42] and others has shown that in developing a representation theory for various classes of linear operations§ among Banach spaces [1] effective use can be made of abstract functions and integrals, just as the general linear functionals over certain B-spaces were earlier discovered to be representable in terms of numerical functions and the integrals of numerical functions. This is especially true for operations defined to a general B-space X from a Lebesgue space, that is, from a space consisting of a class of Lebesgue-integrable numerical functions. To obtain representations for operations of this sort it was found that ready application could be made of various integrals of the Lebesgue type that have been defined for functions taking their values in X.

In the present paper we wish to communicate a representation theory for several types of operators mapping a space $L(S)$, consisting of the real functions that are Lebesgue-integrable over an abstract aggregate S with respect to a fixed class of subsets of S and a fixed measure function [29, 34], into an arbitrary B-space X. The representations will be given in terms of abstract integrals and kernel integrals. The general approach is not new, for it is based on the methods introduced by Gelfand [18] and Dunford [9] to obtain such theorems when S is a bounded real interval. However, in order to extend these known results to the case of an arbitrary S new devices are required since the earlier results were proved by Euclidean methods. In most instances we have been able to make the extension; this has been accomplished by generalizing the Radon-Nikodym theorem [29, 34] to set functions taking their values in an adjoint space and by substituting for differentiation processes the use of convex sets. The class of operators recently introduced by Kakutani [22] and Yosida [44] under the name *weakly completely continu-*

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‡ Numbers in brackets refer to the references at the end. Gelfand's paper [18] is the thesis which he presented in June, 1935.

§ Hereafter for linear operations we shall use the briefer terms *operations* or *operators* since these are the only operations that will come into consideration. An operation, or operator, is thus understood to be a distributive continuous mapping of one B-space into another.

*ous** is also considered. By means of these representation theorems new information is given concerning certain types of operators. This information in turn yields a uniform mean ergodic theorem for weakly c.c.† operations in $L(S)$ and an application to Markoff processes. In addition it provides results which may be of interest in the theory of integral equations. In terms of both abstract integrals and kernel integrals a fairly complete representation theory is given for operations mapping $L(S)$ into the Lebesgue classes $L^q(T)$, $1 \leqq q \leqq \infty$, where T is another aggregate; the types considered are the general, the separable,‡ the weakly c.c., and the c.c. operations. Those results dealing with arbitrary S and T will have as immediate corollaries the corresponding theorems for the sequence spaces l^p;§ the supplying of these corollaries will be left to the reader.

A more precise outline of the contents is perhaps better given by the following table and comments.

CONTENTS

* An operation is weakly completely continuous if it takes bounded sets into weakly compact sets, a set being weakly compact if every infinite subset contains a subsequence converging weakly to some element of the space.

† Hereafter *completely continuous* will be abbreviated to *c.c.*

‡ An operator is *separable* if it maps bounded sets into separable sets. Replacing "separable sets" by "compact sets" furnishes the definition for a *completely continuous* operator, which is necessarily separable. We shall also consider operations that send weakly compact sets into compact sets.

§ Concerning operations on or to a sequence space see [4, 9, 10, 18, 21, 23, 24, 31, 38] and references therein.

Chapter I consists of groundwork. In Part 1 there is a rapid review of Lebesgue integrals for B-space-valued functions, Part 2 is explained by its caption, and Part 3 is chiefly spent in noting that previous results [9] concerning the representation (by kernels and kernel integrals) of measurable and integrable functions in certain function spaces are extensible from Euclidean S to arbitrary S. Most of Chapter II is devoted to establishing for an arbitrary S results known to hold in the Euclidean case; there are occasional refinements. It is believed however that Theorem 2.1.0, the first section of Part 3, and most of Part 4 are new. Chapter III is probably the most important. It includes the rather interesting result that when S is Euclidean and U is weakly c.c. in $L(S)$ the operation U^2 is c.c. This of course implies that U has a finite-dimensional set of fixed points. In Chapter III, Parts 2 A, C consist of sharpenings and extensions of well known theorems, and Part B contains a summary of our results (except for those in Part 3 of Chapter II) concerning kernel representations of operations on $L(S)$ to $L^q(T)$, $1\leqq q\leqq\infty$. These results enable us to state that several sets of conditions, which are less stringent than those heretofore considered, are sufficient that a kernel operation be c.c. from $L(S)$ to $L^q(T)$.

Chapter III is independent of Chapter II.

CHAPTER I. ABSTRACT FUNCTIONS AND INTEGRALS

PART 1. CLASSES OF MEASURABLE AND INTEGRABLE ABSTRACT FUNCTIONS

A. **Preliminary definitions.** We consider an arbitrary aggregate $S=[s]$, a fixed Borel field $\mathcal{E}=[E]$ of subsets of S with* $S \varepsilon \mathcal{E}$, and a fixed non-nega-

* To use Saks' term, E is an *additive family.*

tive completely additive measure function $\alpha(E)$ defined over $\mathcal{E}$. If we let $\mathcal{E}_B$ denote those $E \in \mathcal{E}$ for which $\alpha(E) < \infty$, a triad $[S, \mathcal{E}, \alpha]$ of this sort will be called a *system* provided that S is *decomposable*, that is, S has a *decomposition* consisting of a denumerable number $\{S_i\}$ of disjoint elements of $\mathcal{E}_B$ such that $S = \sum_i S_i$. Those elements of $\mathcal{E}$ for which $\alpha(E) = 0$ are the *null sets* and are denoted by $\mathcal{E}_0$. It is supposed that every subset of a null set is a null set. When the system $[S, \mathcal{E}, \alpha]$ and the exponent q are fixed, $L^q(S)$ is† the B-space composed of those numerical functions $\phi(s)$ which are defined over S, measurable with respect to $\mathcal{E}$, and have $\|\phi\| = (\int_S |\phi(s)|^q d\alpha)^{1/q} < \infty$ for $q < \infty$ and $\|\phi\| = \text{ess. sup. } |\phi(s)| < \infty$ for $q = \infty$. When $q < \infty$ we note that separability for $L^q(S)$ is equivalent to $\mathcal{E}_B$'s being separable under the metric dist $(E, E') = \alpha(E - E') + \alpha(E' - E)$. Moreover if $q < \infty$ then $f(\phi)$ is a linear functional over $L^q(S)$ if and only if an element ϕ' of $L^{q'}(S)$ exists such that $f(\phi) = \int_S \phi(s)\phi'(s) d\alpha$; the norm $\|f\|$ of f equals $\|\phi'\|_{q'}$ and q' is the exponent conjugate to q.‡ Thus for $1 < q < \infty$ $L^q(S)$ is reflexive§ and hence [16] has a weakly compact unit sphere.

Given a function $x(s)$ defined from the points of S to a B-space X, the set of values assumed on a subset E of S will be denoted by $x(E)$. The function $x(s)$ is *separably-valued* if $x(S)$ is separable in X, *weakly compact-valued* if $x(S)$ is weakly compact, and *compact-valued* when $x(S)$ is compact. Should $x(S - E_0)$ be separable for some null set E_0 we say that $x(s)$ is *almost* (or *essentially*) *separably-valued*; the corresponding definitions for $x(s)$'s being *almost weakly compact-valued* and *almost compact-valued* are evident. More particularly, $x(s)$ is *finitely-valued* if it is constant on each of a finite number of measurable sets E_i with $\sum_i E_i = S$, and it is a *simple function* if it is finitely-valued and if $S - E \in \mathcal{E}_B$ when E is the set over which $x(s)$ vanishes. Finally, if Y is a subset of X, then $x(s)$ is *essentially defined to* Y if $Y \supset x(S - E_0)$ for at least one null set E_0.

In a given B-space X the zero element will be denoted by θ or θ_X. When Y is a subset of X the *span* of Y is the smallest closed linear manifold (c.l.m.) containing Y. Given Y in X and a set Γ in the adjoint X^* of X, Γ is said to be a *determining manifold*‖ *for* Y if a finite constant C exists such that, for each $y \in Y$, $\|y\| = \sup_\gamma |\gamma(y)|$, $\gamma \in \Gamma$, $\|\gamma\| \leq C$. Γ is not required to be linear.

† We shall write $L(S)$ for $L^1(S)$. The elements of $L(S)$ will be called *summable functions.*

‡ When $\alpha(S) < \infty$ a proof of this statement, which is a classical result [1] for Euclidean S, can be found in [10, Theorem 46]; the induction to a general decomposable S is easily made. For the case $q = 1$ Nikodym was apparently the first to extend the classical theorem to functions of an abstract variable [30].

§ Even more, $L^q(S)$ is uniformly convex as shown by Clarkson's proof for Euclidean S, *Uniformly convex spaces*, these Transactions, vol. 40 (1936), pp. 396–414.

‖ When $Y = X$ the set Γ is simply a *determining manifold.*

Finally when a real function $\phi(s)$, or a B-space-valued function $x(s)$ or x_s, is being considered as an element of a class of functions, we shall usually write $\phi(.)$, $x(.)$, or $x.$, following the notation of E. H. Moore.

B. **General measurable and integrable functions.** Let $[S, \mathcal{E}, \alpha]$ be a fixed system and consider an abstract function $x(s)$ defined a.e.† in S and having its values in X. If Γ in some set in X^* and $f(x(s))$ is measurable for each $f \in \Gamma$, we say that $x(s)$ is Γ-*measurable*; for the case $\Gamma = X^*$ a Γ-measurable function is *weakly measurable*. To each p lying between 1 and ∞ inclusive there can be associated the subclass $\mathfrak{L}^p(S)[X, \Gamma]$ of Γ-measurable functions defined by the following condition: $x(.)$ is in $\mathfrak{L}^p(S)[X, \Gamma]$ if and only if $f(x(.)) \in L^p(S)$ for each $f \in \Gamma$. Denoting by p' the exponent conjugate to p, we recall the following fundamental theorem [17; 10, Theorem 49; 18].

Theorem 1.1.1. *If Γ is a c.l.m. and $x(.) \in \mathfrak{L}^p(S)[X, \Gamma]$, a finite constant C exists such that*

$$\left| \int_S f(x(s))\phi'(s)d\alpha \right| \leqq C\cdot\|f\|\cdot\|\phi'\|, \qquad f \in \Gamma,\ \phi' \in L^{p'}(S),$$

C being independent of f and ϕ'. Hence

$$y_{\phi'}(f) \equiv \int_S f(x(s))\phi'(s)d\alpha$$

defines an operation $U(\phi') = y_{\phi'}$ from $L^{p'}(S)$ to Γ^*.

An element $x(.)$ of $\mathfrak{L}^p(S)[X, \Gamma]$ is said to be in the class $\mathfrak{L}_0^p(S)[X, \Gamma]$ if for each $E \in \mathcal{E}_B$ there is a point x_E in X such that

$$f(x_E) = \int_E f(x(s))d\alpha, \qquad f \in \Gamma; \tag{1}$$

when $p=1$ it is supposed that such an x_E exists for each $E \in \mathcal{E}$. An element $x(.)$ of $\mathfrak{L}_0^1(S)[X, \Gamma]$ is a Γ-*integrable* function, and any point x_E satisfying (1) is a Γ-*integral of* $x(.)$ *over* E. If Γ is a determining manifold for X and $x(.)$ is Γ-integrable, then $x(.)$ has for each $E \in \mathcal{E}$ a unique Γ-integral over E called *the* Γ-*integral of* $x(.)$ *over* E. When $\Gamma = X^*$ this unique point is referred to as *the integral of* $x(.)$ *over* E and $x(.)$ is said to be *integrable*. Whenever in connection with a given abstract function $x(.)$ the symbol $\int_E x(s)d\alpha$ occurs devoid of qualification, the implication is that $x(.)$ is integrable and that $\int_E x(s)d\alpha$ stands for the integral of $x(s)$ over E.

The following is a slight extension of Theorem 59 of [10].

† The phrase *almost everywhere*, with its usual meaning of "except possibly on a set of measure zero," will be abbreviated throughout to *a.e.*

THEOREM 1.1.2. *When Γ is a determining c.l.m., $x(.)$ is in $\mathfrak{L}_0^p(S)[X, \Gamma]$ if and only if $x(.)\phi'(.)$ is Γ-integrable for each $\phi' \varepsilon L^{p'}(S)$.*

According to Theorem 1.1.1 if $x(.) \varepsilon \mathfrak{L}_0^p(S)[X, \Gamma]$ there exists a constant C independent of ϕ' and f and such that

$$\left| \int_S f(x(s))\phi'(s)d\alpha \right| \leqq C\cdot\|\phi'\|\cdot\|f\|, \qquad f \varepsilon \Gamma, \phi' \varepsilon L^{p'}(S). \tag{2}$$

At the same time it follows from the definition of $\mathfrak{L}_0^p(S)[X, \Gamma]$ that $x(.)\phi'(.)$ is Γ-integrable for each $\phi' \varepsilon L^{p'}(S)$ which is finitely-valued. Moreover, if for a ϕ' of this sort the Γ-integral of $x(s)\phi'(s)$ over E is $x_{E,\phi'}$, it is clear from (2) that

$$| f(x_{E,\phi'}) | \leqq C\cdot\|\phi'\|\cdot\|f\|, \qquad f \varepsilon \Gamma,$$

and hence, since Γ is a determining manifold (for X),

$$\|x_{E,\phi'}\| \leqq C'\|\phi'\| \tag{3}$$

where C' is finite and independent of ϕ'. In addition $x_{E,\phi'}$ is additive over the finitely-valued elements of $L^{p'}(S)$ due to Γ's being determining. Now let $\{\phi_n'\}$ in $L^{p'}(S)$ be finitely-valued elements converging to an arbitrary $\phi' \varepsilon L^{p'}(S)$. It is clear from (3) that $\lim_{m,n\to\infty} \|x_{E,\phi_m'} - x_{E,\phi_n'}\| = \lim_{m,n}\|x_{E,\phi_m'-\phi_n'}\| = 0$. Let $x_{E,\phi'} = \lim_n x_{E,\phi_n'}$. Then

$$f(x_{E,\phi'}) = \lim_n f(x_{E,\phi_n'}) = \lim_n \int_E f(x(s))\phi_n'(s)d\alpha = \int_E f(x(s))\phi'(s)d\alpha, \qquad f \varepsilon \Gamma,$$

so that (1) is satisfied. Since $E \varepsilon \mathcal{E}$ was arbitrary, $x(.)\phi'(.)$ is Γ-integrable.

If on the other hand $x(.)\phi'(.)$ is Γ-integrable for every $\phi' \varepsilon L^{p'}(S)$, then for each $f \varepsilon \Gamma$ the function $f(x(s))\phi'(s)$ is summable for every $\phi' \varepsilon L^{p'}(S)$. Hence $f(x(.)) \varepsilon L^p(S)$ for each $f \varepsilon \Gamma$ [10, Theorem 46]. To see that $x(.) \varepsilon \mathfrak{L}_0^p(S)[X, \Gamma]$ we have only to observe that $x(.)\phi'(.)$ is Γ-integrable whenever ϕ' is the characteristic function of an element $E \varepsilon \mathcal{E}_B$ (or $\mathcal{E}$ if $p=1$).

THEOREM 1.1.3. *If Γ is a determining c.l.m. and $x(.)$ is in $\mathfrak{L}_0^p(S)[X, \Gamma]$, then $U(\phi') = x_{\phi'}$, where $x_{\phi'}$ is the Γ-integral of $x(.)\phi'(.)$ over S, is an operation defined from $L^{p'}(S)$ to X. The mapping $V(f) = f(x(.))$ is an operation from Γ to $L^p(S)$ that coincides over Γ with the adjoint of U.*

From Theorem 1.1.2 and (2) the Γ-integral $x_{\phi'}$ exists and satisfies the inequality

$$| f(x_{\phi'}) | = \left| \int_S f(x(s))\phi'(s)d\alpha \right| \leqq C\cdot\|\phi'\|\cdot\|f\|, \qquad f \varepsilon \Gamma, \phi' \varepsilon L^{p'}(S), \tag{4}$$

where C is independent of ϕ' and f. The mapping V is obviously defined from Γ to $L^p(S)$ and is additive; it is therefore linear since from (4)

$$\|V(f)\| = \|f(x(.))\| \leqq C \cdot \|f\|, \qquad f \in \Gamma,$$

where C is independent of f. From (4) it also follows, since Γ is determining, that

$$\|x_{\phi'}\| \leqq C' \cdot \|\phi'\|, \qquad \phi' \in L^{p'}(S),$$

where C' is independent of ϕ'. Thus the additive mapping U from $L^{p'}(S)$ to X is linear. Finally, from the equality $f(U(\phi')) = \int_S f(x(s))\phi'(s)d\alpha$, $f \in \Gamma$, $\phi' \in L^{p'}(S)$, it is seen that V coincides over Γ with the adjoint of U.

If X is the adjoint Y^* of another space Y, it is evident from Theorem 1.1.1 that any element of $\mathfrak{L}^p(S)[Y^*, Y]$ is also in $\mathfrak{L}_0^p(S)[Y^*, Y]$. The space Y being equivalent to a determining c.l.m. in $X^* = Y^{**}$, Theorem 1.1.3 yields

THEOREM 1.1.4. *If $x(.) \in \mathfrak{L}^p(S)[Y^*, Y]$, then $x(.) \in \mathfrak{L}_0^p(S)[Y^*, Y]$ and the two mappings*

$$V(y) = x(s)(y), \qquad y \in Y,$$

$$U(\phi') = \int_S x(s)\phi'(s)d\alpha, \qquad \phi' \in L^{p'}(S),$$

where $\int_S x(s)\phi'(s)d\alpha$ is the Y-integral of $x(.)\phi'(.)$, are operations defined from Y to $L^p(S)$ and from $L^{p'}(S)$ to Y^* respectively. The operation V coincides over Y with the adjoint of U.

When Y is separable and $p = \infty$, it is possible to be more precise.

THEOREM 1.1.5. *If Y is separable and $x(.) \in \mathfrak{L}^\infty(S)[Y^*, Y]$, then* (i) $x(.) \in \mathfrak{L}_0^\infty(S)[Y^*, Y]$, (ii) *the mappings*

$$V(y) = x(s)(y), \qquad U(\phi') = \int_S x(s)\phi'(s)d\alpha,$$

where $\int_S x(s)\phi'(s)d\alpha$ is the Y-integral of $x(.)\phi'(.)$, are operations from Y to $L^\infty(S)$ and from $L(S)$ to Y^* respectively, (iii) *V coincides over Y with the adjoint of U, and* (iv) ess. sup. $\|x(s)\| = C < \infty$. *The finite constant C is the common norm of U, V, and the functional $F(y, \phi') = \int_S f(x(s))\phi'(s)d\alpha$ bilinear over $Y \times L(S)$.*

Conclusions (i)–(iii) are corollaries of Theorem 1.1.4. The remainder follows from Theorem 9 of [10].

C. **Measurable functions and absolutely integrable functions.** Beginning again from a somewhat different viewpoint, $x(s)$ is *measurable* [3] if it is the limit a.e. of some sequence of simple functions; since S is decomposable this is equivalent to being the limit a.e. of a sequence of finitely-valued functions. For each p in the range $1 \leqq p < \infty$ there exists the subclass of measurable functions having $\int_S \|x(s)\|^p d\alpha < \infty$. Denoting this subclass by $\mathfrak{A}^p(S)[X]$ and letting $\mathfrak{A}^\infty(S)[X]$ be those measurable functions for which ess. sup. $\|x(s)\| < \infty$ it follows, as Garrett Birkhoff has shown [2, 9], that $\mathfrak{A}^p(S)[X] \subset \mathfrak{L}_0^p(S)[X, X^*]$ for $1 \leqq p \leqq \infty$. The elements of $\mathfrak{A}^1(S)[X]$ will be called *absolutely integrable.*

THEOREM 1.1.6. *If $x(.) \in \mathfrak{A}^p(S)[X]$, the function $x(.)\phi'(.)$ is absolutely integrable for each $\phi'(.) \in L^{p'}(S)$ and $U(\phi') = \int_S x(s)\phi'(s)d\alpha$ defines an operation U from $L^{p'}(S)$ to X. When $p = \infty$ the norm of U is $|U| =$*ess. sup. $\|x(s)\|$, *and*

$$|U| \leqq \left(\int_S \|x(s)\|^p d\alpha\right)^{1/p}$$

when $p < \infty$.

The absolute integrability of $x(.)\phi'(.)$ should be evident. The existence and linearity of U are likewise obvious, in view of Theorem 1.1.3 and the inclusion of $\mathfrak{A}^p(S)[X]$ in $\mathfrak{L}_0^p(S)[X, X^*]$. For the case $p = \infty$ it is known [31, 3.11] that the operation $V(f) = f(x(.))$ from X^* to $L^\infty(S)$ has $|V| =$ ess. sup. $\|x(s)\|$; since V is the adjoint of U this means that $|U| =$ ess. sup. $\|x(s)\|$. The concluding inequality, for the case $p < \infty$, has already been proved in Theorem 2.4 of [9].

It is known that the two properties of weak measurability and almost separable valuedness together are sufficient as well as necessary for measurability [18, 31]. This remains true when a less stringent condition is substistituted for weak measurability; and if a third assumption is made, $x(.)$ becomes not only measurable but also essentially bounded.

THEOREM 1.1.7. *For a given $x(.)$ suppose that for some null set E_0 the set $x(S - E_0)$ has in X a separable span Y and that $x(.)$ is Γ-measurable for some manifold Γ which is determining for Y. Then* (i) *$x(.)$ is measurable. If it is also true that* ess. sup.$_s$ $|f(x(s))| < \infty$ *holds for each $f \in \Gamma$ and Γ is closed and linear, then* (ii) *$x(.) \in \mathfrak{A}^\infty(S)[X]$, and hence* (iii) *$x(.)\phi(.)$ is absolutely integrable for each $\phi \in L(S)$ and $U(\phi) = \int_S x(s)\phi(s)d\alpha$ defines as operation U from $L(S)$ to X having $|U| =$* ess. sup. $\|x(s)\|$.

Conclusion (i) has been established in Theorem 2.1 of [32]. If it is also true that Γ is a c.l.m., then, since Y is separable, there is in Γ a separable c.l.m. Γ' which is determining for Y. Thus if $x(.) \in \mathfrak{L}^\infty(S)[X, \Gamma]$ clearly $x(.) \in \mathfrak{L}^\infty(S)[Y, \Gamma']$; from Theorem 9 of [10] it follows that ess. sup. $\|x(s)\|$

$<\infty$, so that the measurable function $x(.)$ is in $\mathfrak{A}^\infty(S)[X]$. Conclusion (iii) results from (ii) and Theorem 1.1.6.

PART 2. SOME CONVEX SETS RELATED TO INTEGRABLE FUNCTIONS

This part will consist of a few results involving inclusion relationships between certain closed convex sets containing the functional values of an integrable function and other closed convex sets associated with the function's indefinite integral. In a B-space X the closed convex hull of a subset Y will be denoted by $C[Y]$. For a fixed function $x(.)$ which is integrable over every $E \varepsilon \mathcal{E}_B$ the set of all quotients $(\int_E x(s)d\alpha)/\alpha(E)$ with $0<\alpha(E)<\infty$ will be represented by J. Thus J is defined if $x(.) \varepsilon \mathfrak{L}_0^p(S)[X, X^*]$, $1 \leqq p \leqq \infty$.

THEOREM 1.2.1. *If x_0 is a point disjoint with a closed convex set X', there exist an $f_0 \varepsilon X^*$ and a constant c such that $f_0(x_0)>c$ while $f_0(x)\leqq c$ for $x \varepsilon X'$.*

By a theorem of Mazur [27, p. 80] there is a closed convex body K disjoint with x_0 and containing X'. There must then be a nondegenerate closed sphere N about x_0 such that NK is vacuous. From a theorem of Eidelheit [12] the disjunction of the two closed convex bodies N and K implies the existence of an $f_0 \varepsilon X^*$ and a constant c such that $\|f_0\| \neq 0$, $f_0(x) \geqq c$ for $x \varepsilon N$, and $f_0(x) \leqq c$ for $x \varepsilon K$. Since $\|f_0\| \neq 0$ and no nondegenerate sphere can be contained in a hyperplane, there is an $x_1 \varepsilon N$ failing to satisfy the equation $f_0(x)=c$; thus $f_0(x_1)=c+\delta$ where $\delta>0$. If now $f_0(x_0)=c$, it follows that $f_0(2x_0-x_1)=c-\delta<c$ where $2x_0-x_1$ is in N. This contradicts the inequality $f_0(x)\geqq c$ for $x \varepsilon N$, and hence $f_0(x_0)>c$ must hold.

THEOREM 1.2.2. *If $x(.)$ is integrable over every $E \varepsilon \mathcal{E}_B$, then $C[J] \subset C[x(S-E_0)]$ for each $E_0 \varepsilon \mathcal{E}_0$.*

Let $x_E=\int_E x(s)d\alpha$, $E \varepsilon \mathcal{E}_B$. If there were an $E \varepsilon \mathcal{E}_B-\mathcal{E}_0$ for which $x_E/\alpha(E) \notin C[x(S-E_0)]$, then by the preceding theorem a constant c and an $f_0 \varepsilon X^*$ would exist such that $f_0(x_E/\alpha(E))>c$ while $x \varepsilon C[x(S-E_0)]$ implies $f_0(x)\leqq c$. Since $\alpha(E_0)=0$ we would then have $f_0(x(s))\leqq c$ a.e. in E, so that $\int_E f_0(x(s))d\alpha \leqq c\alpha(E)$. Yet $\int_E f_0(x(s))d\alpha=f_0(x_E)>c\alpha(E)$, which is a contradiction. Thus $J \subset C[x(S-E_0)]$ and hence $C[J]\subset C[x(S-E_0)]$.

An immediate consequence of Theorem 1.2.2 is

THEOREM 1.2.3. *For a function $x(.)$ integrable over every $E \varepsilon \mathcal{E}_B$ the inclusion $C[J] \subset \prod C[x(S-E_0)]$ holds, the product being taken as E_0 varies over $\mathcal{E}_0$.*

Another easy corollary is given in

THEOREM 1.2.4. *If $x(.) \varepsilon \mathfrak{L}_0^p(S)[X, X^*]$ and Y is a c.l.m. containing $x(S-E_0)$ for some $E_0 \varepsilon \mathcal{E}_0$ then $Y \ni U(\phi')=\int_S x(s)\phi'(s)d\alpha$ for every $\phi' \varepsilon L^{p'}(S)$,* *that is,* $Y \supset U(L^{p'}(S))$. *Thus* U *is a separable operation when* $x(.)$ *is almost separably-valued.*

Since Y is a closed convex set containing $x(S-E_0)$, Theorem 1.2.2 implies that $Y \supset J$. From the linearity of Y it then follows that $Y \ni \int_E x(s)d\alpha$ for each $E \in \mathcal{E}_B$. If $p=1$ the complete additivity of $\int_E x(s)d\alpha$ over $\mathcal{E}$ [31, 10] and the fact that Y is closed and linear yield the conclusion that $Y \ni \int_E x(s)d\alpha$ for every $E \in \mathcal{E}$. The simple functions being dense in $L^{p'}(S)$ for $p'<\infty$ and the finitely-valued ones being dense in $L^\infty(S)$, it follows from the linearity of U that $U(\phi')$ is in the c.l.m. Y for each $\phi' \in L^{p'}(S)$.

Corollary. *If* $x(.) \in \mathfrak{A}^p(S)[X]$, *then* $U(\phi')=\int_S x(s)\phi'(s)d\alpha$ *is a separable operation from* $L^{p'}(S)$ *to* X.

Theorem 1.2.4 can be given a considerably sharper form when $p=\infty$. For this purpose let $R(x)=-x$ for $x \in X$, and let $\mathfrak{P}$ be the essentially non-negative elements of $L(S)$. The subset of $\mathfrak{P}$ for which $\|\phi\|=1$ holds will be denoted by $\mathfrak{P}_1$.

Theorem 1.2.5. *Suppose that* $x(.)$ *is in* $\mathfrak{L}_0^\infty(S)[X, X^*]$ *and that* Y *is a closed convex set containing* $x(S-E_0)$ *for some* $E_0 \in \mathcal{E}_0$. *For* $U(\phi)=\int_S x(s)\phi(s)d\alpha$, $\phi \in L(S)$, *it follows that*

(i) $U(\phi) \in C[x(S-E_0)] \subset Y$ *when* $\phi \in \mathfrak{P}_1$,

(ii) *if* $Y \ni \theta$ *then* $U(\phi) \in Y$ *for each* $\phi \in \mathfrak{P}$ *having* $\|\phi\| \leqq 1$,

(iii) *if* $Y \supset R(x(S-E_0))$ *then* $U(\phi) \in Y$ *for each* ϕ *with* $\|\phi\| \leqq 1$,

(iv) *if* Y *is linear then* $U(\phi) \in Y$ *for every* ϕ.

In (i) the second relationship is obvious. To obtain the first it is sufficient to show that $Y \ni U(\phi)$ whenever ϕ is an essentially non-negative simple function having $\|\phi\|=1$. For such a ϕ we have $U(\phi)=\sum_1^n \phi_i \int_{E_i} x(s)d\alpha=\sum_1^n \alpha_i x_i$ where $\alpha_i \equiv \phi_i \alpha(E_i) \geqq 0$, $\infty > \alpha(E_i) > 0$, and $x_i=1/\alpha(E_i) \cdot \int_{E_i} x(s)d\alpha$. Since $x_i \in J$, $\alpha_i \geqq 0$, and $\sum_1^n \alpha_i = \|\phi\|=1$, it follows that $U(\phi) \in C[J]$ and hence (Theorem 1.2.2) $U(\phi) \in C[x(S-E_0)]$. Statement (ii) results immediately from (i), the homogeneity of U, and the convexity of Y. Conclusion (iii) is also implied by (i). For an arbitrary $\phi \in L(S)$ can be written as $\phi=\phi_1-\phi_2$ where $\phi_i \in \mathfrak{P}$ and $\|\phi\|=\sum_1^2\|\phi_i\|$. Thus

$$U(\phi/\|\phi\|) = \sum_1^2 \rho_i x_i, \qquad \|\phi\| > 0,$$

where $\rho_i=\|\phi_i\|/\|\phi\| \geqq 0$, $x_1=U(\phi_1/\|\phi_1\|)$, and $x_2=-U(\phi_2/\|\phi_2\|)$. Since (i) implies that $x_1 \in C[x(S-E_0)]$ and $x_2 \in R(C[x(S-E_0)])$, and since $\rho_1+\rho_2=1$ where $\rho_i \geqq 0$, it is clear that $Y \ni U(\phi/\|\phi\|)$ due to the convexity of Y. We can then conclude, Y being a convex set containing θ, that $Y \ni U(\phi)$ for every ϕ with $\|\phi\| \leqq 1$. The final statement, (iv), results from either (iii) or Theorem 1.2.4.

THEOREM 1.2.6. *Suppose that for* $x(.) \varepsilon \mathfrak{L}_0^\infty(S)[X, X^*]$ *there exist a null set* E_0, *a closed convex set* Y, *and a constant* C *such that* $Y \supset x(S-E_0)$ *and* $\|y\| = C$ *for each* $y \varepsilon Y$. *Then* $U(\phi) \varepsilon Y$ *for every* $\phi \varepsilon \mathfrak{P}_1$ *and hence* $\|U(\phi)\| = C\|\phi\|$ *for every* $\phi \varepsilon \mathfrak{P}$.

It will now be shown that for a restricted $x(s)$ the inclusion reverse to that of Theorem 1.2.2 holds for at least one $E_0 \varepsilon \mathcal{E}_0$.

THEOREM 1.2.7. *If* $x(.)$ *is integrable over every* $E \varepsilon \mathcal{E}_B$ *and is also measurable, there is an* $E_0' \varepsilon \mathcal{E}_0$ *such that* $C[x(S-E_0')] \subset C[J]$.

Since $x(.)$ is measurable, there is a null set E'' such that $x(S-E'')$ is separable. Thus if $x(.)$ is redefined so as to vanish over E'', the new function $x'(.)$ will be measurable and $x_E' \equiv \int_E x'(s)d\alpha = \int_E x(s)d\alpha$ for every $E \varepsilon \mathcal{E}_B$. Moreover $x'(S)$ will be a separable set in X. If in $\mathcal{E}_0$ there is an E' with the property that $C[x'(S-E')] \subset C[J']$, it will then follow, should the theorem be true for $x'(.)$, that $C[x(S-E''-E')] \subset C[x'(S-E')] \subset C[J'] = C[J]$, where $E_0' = E''+E' \varepsilon \mathcal{E}_0$. Hence we may suppose that $x(S)$ is separable. From the measurability of $x(.)$ it can be concluded that $x^{-1}(Y)$ is in $\mathcal{E}$ for any open sphere Y in X; since $x(S)$ is a separable set in X this remains true if Y is an arbitrary open set and hence if Y is any Borel-measurable set. It is now evident, $C[J]$ being closed, that the set $E' = x^{-1}(X - C[J])$ must be measurable. To prove the theorem we have only to show that $\alpha(E') = 0$.

Each point $x_0 \varepsilon x(E')$ is disjoint with $C[J]$; to each such x_0 there corresponds by Theorem 1.2.1 an element f of X^* and a constant c such that $f(x_0) > c$ and $f(x) \leqq c$ for $x \varepsilon C[J]$. From the continuity of f an open sphere N about x_0 exists with the property that $f(x) > c$ holds for every $x \varepsilon N$. Let each point of $x(E')$ be covered by a sphere of this kind. Since $x(E')$ is a separable set in X, a denumerable number N_m, $m = 1, 2, \cdots$, of these spheres suffice to cover $x(E')$. Letting $E_m = x^{-1}(N_m)$ it is clear that E_m is measurable and that $\sum_1^\infty E_m \supset E'$. Thus if $\alpha(E')$ is positive, then $\alpha(E_{m_0}) > 0$ holds for some m_0. Taking a decomposition $\{S_n\}$ of S, the inequality $\infty > \alpha(S_n E_{m_0}) > 0$ must be true for at least one n. Letting E^* be the set $S_n E_{m_0}$ for one such n, obviously $0 < \alpha(E^*) < \infty$ and $x_{E^*}/\alpha(E^*) \varepsilon J$. From the definitions of N_{m_0}, E_{m_0}, and E^* there is an $f_0 \varepsilon X^*$ and a constant c_0 such that $f_0(x(s)) > c$ for every $s \varepsilon E^*$, while $f_0(x) \leqq c_0$ when $x \varepsilon C[J]$. Due to the inequality $\infty > \alpha(E^*) > 0$ we then have $\int_{E^*} f_0(x(s))d\alpha > c_0\alpha(E^*)$ and at the same time $\int_{E^*} f_0(x(s))d\alpha = f_0(x_{E^*}) \leqq c_0\alpha(E^*)$. This contradiction ends the proof since it implies $\alpha(E') = 0$.

Combining Theorems 1.2.3 and 1.2.7 gives us

THEOREM 1.2.8. *If $x(.)$ is integrable over every $E \in \mathcal{E}_B$ and is measurable, then an $E_0' \in \mathcal{E}_0$ exists such that*

$$C[x(S - E_0')] = C[J] = \prod C[x(S - E_0)],$$

the product being taken as E_0 varies over $\mathcal{E}_0$.

For measurable functions Theorem 1.2.4 may be rounded out as follows.

THEOREM 1.2.9. *Let $x(.) \in \mathfrak{L}_0^p(S)[X, X^*]$ be measurable. For a c.l.m. Y the following are equivalent*: (i) *$x(.)$ is essentially defined to Y*, (ii) *$Y \ni \int_E x(s)d\alpha$ for every $E \in \mathcal{E}_B$, and* (iii) *$Y \ni U(\phi) = \int_S x(s)\phi(s)d\alpha$ for every $\phi \in L^{p'}(S)$.*

For (iii) clearly implies (ii), and (iii) follows from (i) by Theorem 1.2.4. If (ii) holds, then $Y \supset C[J]$, and hence $Y \supset x(S-E_0')$ for some $E_0' \in \mathcal{E}_0$ according to Theorem 1.2.8.

When $p = \infty$ more precise results can be obtained.

THEOREM 1.2.10. *Suppose $x(.)$ in $\mathfrak{L}_0^\infty(S)[X, X^*]$ is measurable. Then* (I) *an $E_0' \in \mathcal{E}_0$ exists such that*

$$\prod_{E_0 \in \mathcal{E}_0} C[x(S - E_0)] = C[x(S - E_0')] = C[J] = C[U(\mathfrak{P}_1)].$$

Let Y be a closed convex set. Then (II) *these three conditions are equivalent*:

(i) *$Y \supset x(S-E_0)$ for some $E_0 \in \mathcal{E}_0$,*

(ii) *$Y \ni 1/\alpha(E) \cdot \int_E x(s)d\alpha$ for every $E \in \mathcal{E}_B - \mathcal{E}_0$,*

(iii) *$Y \ni U(\phi) = \int_S x(s)\phi(s)d\alpha$ for every $\phi \in \mathfrak{P}_1$;*

(III) *these three are equivalent*:

(i) *Y satisfies* (II)(i) *and $Y \ni \theta$,*

(ii) *Y satisfies* (II)(ii) *and $Y \ni \theta$,*

(iii) *$Y \ni U(\phi)$ for every $\phi \in \mathfrak{P}$ with $\|\phi\| \leqq 1$;*

and (IV) *these are equivalent*:

(i) *for some $E_0 \in \mathcal{E}_0$ both $x(S-E_0)$ and $R(x(S-E_0))$ are in Y,*

(ii) *$Y \ni (\delta/\alpha(E))\int_E x(s)d\alpha$ for every $E \in \mathcal{E}_B - \mathcal{E}_0$ and every $\delta = \pm 1$,*

(iii) *$Y \ni U(\phi)$ for each ϕ with $\|\phi\| \leqq 1$.*

(V) *Finally, for a given constant K these are equivalent*:

(i) *$\|U(\phi)\| = K$ for every $\phi \in \mathfrak{P}_1$,*

(ii) *an $E_0' \in \mathcal{E}_0$ exists such that $\|x\| = K$ for $x \in C[x(S-E_0')]$.*

Hence if (V)(i) *or* (V)(ii) *holds, then $\|x(s)\| = K$ a.e. and $\|\int_E x(s)d\alpha\| = K\alpha(E)$ for every $E \in \mathcal{E}_B$.*

(I) follows immediately from Theorem 1.2.8, conclusion (i) of Theorem 1.2.5, and the fact that $J \subset U(\mathfrak{P}_1)$. (II) and (III) are evident consequences of (I) and (II) respectively. (IV) results from (III) and Theorem 1.2.5 on considering the operation $U'(\phi) = \int_S(-x(s))\phi(s)d\alpha = -U(\phi)$. The equivalence of the two conditions in (V) follows from (I) and the fact that $C[U(\mathfrak{P}_1)]$ is simply the closure of the convex set $U(\mathfrak{P}_1)$.

Example 3 in Birkhoff's paper [2] shows that Theorems 1.2.7–1.2.10 all become false if the hypothesis of measurability for $x(.)$ be dropped.

PART 3. KERNEL REPRESENTATIONS OF ABSTRACT FUNCTIONS AND INTEGRALS

We now consider a second system $[T, \mathcal{J}, \beta]$ entirely analogous to $[S, \mathcal{E}, \alpha]$, with $\mathcal{J}_0$ denoting all those elements of $\mathcal{J}$ having β-measure zero and $\mathcal{J}_B$ all those having finite β-measure.

A. **Measurable functions in terms of numerical kernels.** When the range space X consists of numerical functions defined and measurable over T, it sometimes happens that each measurable function $x(s)$ having its values in X can be represented by means of a numerical kernel defined and measurable* over $S\times T$. The first theorem below gives a set of conditions on X that are sufficient for the existence of such representations.

THEOREM 1.3.1. *Suppose X satisfies these conditions*:

(a) *if $\psi^m \in X$, $m=1, 2, \cdots$, and $\lim_m \|\psi^m\|=0$, then $\{\psi^m(t)\}$ converges to zero in measure over every $F \in \mathcal{J}_B$,*

(b) *$\psi(.) \in X$ implies $\|\psi\|=0$ if and only if $\psi(t)=0$ a.e.,*

(c) *if $\{T_i\}$ is a decomposition of T and with each $\psi(.) \in X$ are associated the functions $\psi_n(t)=\psi(t)\phi_n(t)$, $n=1, 2, \cdots$, where $\phi_n(t)$ is the characteristic function of $\sum_1^n T_i$, then*

(i) *$\psi_n(.) \in X$ for each n,*

(ii) *$\lim_m \psi^m=\psi$ in X implies $\lim_m \psi_n^m=\psi_n$ in X for each n.*

Under these conditions it follows that if x_s is measurable from S to X there is a numerical kernel $K(s, t)$ measurable over $S\times T$ and such that for almost† *every $s \in S$ the equality $x_s(t)=K(s, t)$ holds a.e. in T.*‡ *Hence if* (b) *is strengthened to*

(b′) *$\psi(t)$ is measurable and $\psi(t)=0$ almost everywhere if and only if $\psi(.) \in X$ and $\|\psi\|=0$,*

then for almost every s $K(s, .)$ is in X and coincides with the point x_s.

The demonstration of this theorem for the case $\alpha(S)<\infty$ and $\beta(T)<\infty$ has been given in 3.1 of [9].§ The extension to the case of a general decomposable T follows from a reapplication of the same methods. The final step, to a general decomposable S, is then easily made.

* That is, measurable with respect to the Borel field of subsets of $S\times T$ determined by the two fields E and F [33].

† If x_s is defined over all of S the word "almost" may be deleted.

‡ $K(s, t)$ is then said to be a *measurable representation of* $x(s)$.

§ The proof given there is for Euclidean S and T but can be carried over to the present case without change.

Since $L^q(T)$ satisfies conditions (a), (b′), and (c) for $1 \leqq q \leqq \infty$ we have [9]

THEOREM 1.3.2. *If $x(s)$ is measurable from S to $L^q(T)$, a measurable kernel $K(s, t)$ exists over $S\times T$ such that $K(s, .)=x(s)$ in $L^q(T)$ a.e. in S. Any two such representations of $x(.)$ differ over $S\times T$ on at most a set of measure zero.*

On the other hand the converse (partial when $q=\infty$) of Theorem 1.3.2 also holds.

THEOREM 1.3.3. *Let $K(s, t)$ be a kernel over $S\times T$ and let q be fixed. Suppose that*

(i) *$K(s, t)$ is measurable,*

(ii) *$K(s, .) \in L^q(T)$ for almost every s, and*

(iii) *if $q=\infty$ then $K(s, .)$ is almost separably-valued in $L^q(T)$.*

Then $x(s)=K(s, .)$ as a function defined to $L^q(T)$ is almost separably-valued and is measurable.

We first show that $K(s, .)$ is almost separably-valued when $q<\infty$. Suppose $K(s, t)$ is the characteristic function of a set having finite measure in the product space $S\times T$. There is then a sequence $\{K_n(s, t)\}$ such that each $K_n(s, t)$ is the characteristic function of a finite sum of sets of the form $E\times F$ and

$$\lim_n \iint_{S\times T} | K_n(s, t) - K(s, t) |^q d(\alpha \times \beta) = 0.$$

Thus there is a subsequence $\{K_{n_i}\}$ such that

$$\lim_i \int_T | K_{n_i}(s, t) - K(s, t) |^q d\beta = 0$$

a.e. in S. This shows that the values of $x(s)=K(s, .)$ lie for almost all s in a separable subset of $L^q(T)$. If K is the characteristic function of a set of infinite measure in the product space, then $K=\sum_n \phi_n K$ where ϕ_n is the characteristic function of G_n and the sequence $\{G_n\}$ forms a partition of $S\times T$ into sets of finite measure. So in this case also we have $x(.)$ almost separably-valued. Thus the abstract function $x(s)=K(s, .)$ corresponding to a finitely-valued kernel $K(s, t)$ is almost separably-valued. Now suppose $K(s, t)$ is an arbitrary function satisfying (i) and (ii) for $q<\infty$. Since K is measurable on $S\times T$ there is a sequence $\{K_n(s, t)\}$ of finitely-valued functions having the properties

$$\lim_n K_n(s, t) = K(s, t) \qquad \text{a.e. on } S \times T, \tag{1}$$

$$| K_n(s, t) - K(s, t) | \leqq | K(s, t) | \qquad \text{on } S \times T. \tag{2}$$

The Fubini theorem (1) shows that, for almost every s, $\lim_n K_n(s, t) = K(s, t)$ a.e. in T. This fact together with (2), (ii), and the Lebesgue convergence theorem yields for almost every s

$$\lim_n \int_T | K_n(s, t) - K(s, t) |^q d\beta = 0,$$

which proves that $x(s) = K(s, .)$ is almost separably-valued.

Thus $x(s) = K(s, .)$ is almost separably-valued for $1 \leqq q \leqq \infty$. Since the measurability of K implies that $\int_T K(s, t)\psi'(t)d\beta$ is measurable in s for each $\psi'(.) \varepsilon L^{q'}(T)$, it then follows from Theorem 1.1.7 that $x(.)$ is measurable.

B. **The representation of abstract integrals by kernel integrals.** We recall the following two theorems giving representations for certain integrals. These have been obtained previously (for Euclidean S and T) in [31] and [9].

THEOREM 1.3.4. *Suppose $x(.)$ in $\mathfrak{L}_0^{p'}(S)[L^q(T), L^{q'}(T)]$ has a measurable representation $K(s, t)$. A necessary and sufficient condition that the operation* $U(\phi) = \int_S x(s)\phi(s)d\alpha$ from $L^p(S)$ to $L^q(T)$ be expressible as*

$$U(\phi) = \int_S K(s, t)\phi(s)d\alpha$$

is that $\phi \varepsilon L^p(S)$ and $\psi' \varepsilon L^{q'}(T)$ imply that $\int_T \psi'(t)\{\int_S K(s, t)\phi(s)d\alpha\}d\beta = \int_S \phi(s)\{\int_T K(s, t)\psi'(t)d\beta\}d\alpha$ finitely.

This is merely Theorem 7.3 of [31] carried over to the present more general S and T; the proof remains precisely the same.

From Theorems 1.3.2 and 1.3.4 we can now derive

THEOREM 1.3.5 [9]. *Suppose $x(.)$ is in $\mathfrak{A}^{p'}(S)[L^q(T)]$. Then a kernel $K(s, t)$ exists with these properties:*

(i) *$K(s, t)$ is measurable, and*

(ii) *$K(s, .) = x(s)$ in $L^q(T)$ for almost every s.*

If $K(s, t)$ is any kernel satisfying (i) *and* (ii), *then*

(iii) $(\int_S \|K(s, .)\|^{p'} d\alpha)^{1/p'} \equiv C < \infty$ *when* $p' < \infty$ *and* ess. sup. $\|K(s, .)\| \equiv C < \infty$ *when* $p' = \infty$,

(iv) *the separable operation $U(\phi) = \int_S x(s)\phi(s)d\alpha$ can be written as $U(\phi) = \int_S K(s, t)\phi(s)d\alpha$, $\phi \varepsilon L^p(S)$,*

(v) $|U| \leqq C$ *for* $p' < \infty$ *and* $|U| = C$ *for* $p' = \infty$.

The existence of a $K(s, t)$ satisfying (i) and (ii) comes immediately from Theorem 1.3.2. Property (iii) is obvious in view of (ii) and the fact that $x(.) \varepsilon \mathfrak{A}^{p'}(S)[L^q(T)]$. In (iv) the operation U exists and is separable by Theo-

* When $q = \infty$, this integral is an $L(T)$-integral.

rem 1.1.6 and the corollary to Theorem 1.2.4. Moreover if $K(s, t)$ satisfies (i) and (ii) we have for each $\psi'(.) \varepsilon L^{q'}(T)$ that

$$\int_T |K(s,t)| \cdot |\psi'(t)| d\beta \leqq \|\psi'\| \cdot \|x(s)\|,$$

by Hölder's inequality. Since $\|x(.)\| \varepsilon L^{p'}(S)$ this means that $\int_S |\phi(s)| \cdot \{\int_T |K(s, t)| \cdot |\psi'(t)| d\beta\} d\alpha$ exists finitely whenever $\phi(.) \varepsilon L^p(S)$ and $\psi'(.) \varepsilon L^{q'}(T)$. Since $K(s, t)\phi(s)\psi'(t)$ is measurable, a classical theorem of Tonelli implies that the two repeated integrals in Theorem 1.3.4 exist finitely and are equal. From the latter theorem (iv) then follows. Finally, (v) results from (ii) and Theorem 1.1.6.

If in Theorem 1.3.5 we begin with a kernel instead of an abstract function, we can state

THEOREM 1.3.6. *Let $K(s, t)$ be a kernel satisfying for a fixed q conditions* (i)–(iii) *of Theorem* 1.3.3 *and for a fixed p' condition* (iii) *of Theorem* 1.3.5. *Then $x(s) = K(s, .)$ is in $\mathfrak{A}^{p'}(S)[L^q(T)]$ and $U(\phi) = \int_S K(s, t)\phi(s) d\alpha$ is an operation defined and separable from $L^p(S)$ to $L^q(T)$, with $|U| \leqq C$ when $p' < \infty$ and $|U| = C$ when $p' = \infty$.*

It is clear from Theorem 1.3.3 and property (iii) of Theorem 1.3.5 that $x(s) = K(s, .)$ is in $\mathfrak{A}^{p'}(S)[L^q(T)]$. The remaining conclusions follow from Theorem 1.3.5.

Later it will be seen that the operation U in Theorem 1.3.6 is c.c. when $p' < \infty$ [38] (Theorems 3.2.11 and 3.2.12) and that U takes weakly compact sets in $L(S)$ into compact sets in $L^q(T)$ when $p' = \infty$ (Theorem 3.2.1).

For the case $p' = \infty$ it results from Theorems 1.3.3, 1.1.7, and 1.3.6 that the fourth assumption in the last theorem can be weakened as follows.

THEOREM 1.3.7. *Suppose $K(s, t)$ satisfies conditions* (i)–(iii) *in Theorem* 1.3.3 *and in addition* ess. sup.$_s$ $|\int_T K(s, t)\psi'(t) d\beta| < \infty$ *is true for each $\psi'(.) \varepsilon L^{q'}(T)$. Then $x(s) = K(s, .)$ is in $\mathfrak{A}^\infty(S)[L^q(T)]$ and the operation $U(\phi) = \int_S K(s, t)\phi(s) d\alpha$ is defined and separable from $L(S)$ to $L^q(T)$ with $|U| =$* ess. sup.$_s$ $(\int_T |K(s, t)|^q d\beta)^{1/q}$ *for $q < \infty$ and $|U| =$* ess. sup.$_{s,t}$ $|K(s, t)|$ *for $q = \infty$.*

CHAPTER II. GENERAL OPERATIONS TO RESTRICTED SPACES

PART 1. THE REPRESENTATION IN ADJOINT SPACES OF SET FUNCTIONS AND LINEAR OPERATIONS BY ABSTRACT INTEGRALS

A. **Integral representations of absolutely continuous and Lipschitzean set functions.** The first theorem below may be regarded either as a generalization (to the case of an abstract S) of a result of Gelfand [18] or as an extension of the Radon-Nikodym theorem [29, 34] from real-valued set functions to those taking their values in an adjoint space. Two proofs of the theorem will be given.

THEOREM 2.1.0. *Let $X=[x]$ be the adjoint of another B-space $Y=[y]$ and let x_E be defined from $\mathcal{E}_B$ to X. Suppose that Y' is a separable linear subset of Y and that*

(i) *for each $y \varepsilon Y'$ the real-valued set function $x_E(y)$ is completely additive over* $\mathcal{E}(E')$ for every $E' \varepsilon \mathcal{E}_B$,*

(ii) *$x_E(y)=0$ when E is a null set and $y \varepsilon Y'$, and*

(iii) *the numerical function*

$$\sigma_E \equiv \sup_y \frac{1}{\|y\|} | x_E(y) |, \qquad y \varepsilon Y', \ y \neq \theta,$$

which is defined over $\mathcal{E}_B$ has $\nu_{E'}$, its total variation over E', finite for every $E' \varepsilon \mathcal{E}_B$.

Then there exists an x_s defined from S to X such that

$$(2.1.01) \qquad x_E(y) = \int_E x_s(y) d\alpha, \qquad E \varepsilon \mathcal{E}_B, \ y \varepsilon Y',$$

$$(2.1.02) \qquad \nu_E = \int_E \|x_s\| d\alpha \qquad \textit{for every } E \varepsilon \mathcal{E}_B.$$

First proof. The initial step† is to show that if $E \varepsilon \mathcal{E}_B$ and $\{E_j\}$ is a finite or denumerable number of disjoint elements of $\mathcal{E}$ with $\sum_j E_j = E$ then

$$(1) \qquad \nu_E = \sum_j \nu_{E_j}.$$

Since it is obvious from the definition of ν_E that $\nu_E \geqq \sum_j \nu_{E_j}$, it will be sufficient to establish the reverse inequality

$$(2) \qquad \nu_E \leqq \sum_j \nu_{E_j}.$$

If we choose disjoint measurable sets $E_1', \cdots, E_n'$ in E such that $\nu_E - \epsilon < \sum_1^n \sigma_{E_i'}$ where $\epsilon > 0$, clearly there are elements $y_1, \cdots, y_n$ in Y' for which

$$\nu_E - \epsilon < \sum_{i=1}^n \rho_i | x_{E_i'}(y_i) |,$$

where $\rho_i \|y_i\| = 1$. Since (i) implies that $x_{E_i'}(y_i) = \sum_j x_{E_i' E_j}(y_i)$ and $\sum_j | x_{E_i' E_j}(y_i) |$

* For a fixed $E' \varepsilon \mathcal{E}$ the symbol $\mathcal{E}(E')$ denotes the Borel field composed of all sets of the form EE' where $E \varepsilon \mathcal{E}$.

† The theorem is trivial when Y' is vacuous or consists of the zero element.

$<\infty$ for each i, we can conclude that

$$\nu_E - \epsilon < \sum_{i=1}^{n} \rho_i \left| \sum_j x_{E_i'E_j}(y_i) \right| < \sum_j \sum_i \rho_i \left| x_{E_i'E_j}(y_i) \right|$$

and hence that $\nu_E - \epsilon < \sum_j \sum_i \sigma_{E_i'E_j} \leqq \sum_j \nu_{E_j}$ whenever $\epsilon > 0$. This vindicates (2), so that (1) holds and ν_E is completely additive over $\mathcal{E}(E')$ for each $E' \varepsilon \mathcal{E}_B$. An additional property of ν_E is that $\nu_E = 0$ when E is a null set, by virtue of (ii). In view of the decomposability of S and the Radon-Nikodym theorem these two properties lead to the existence of a measurable non-negative function $\phi_0(s)$ satisfying the condition that

$$\text{(3)} \qquad \int_{E'} \phi_0(s) d\alpha = \nu_{E'} \qquad \text{for every } E' \varepsilon \mathcal{E}_B.$$

S being decomposable it is also true that

$$\text{(4)} \qquad | \phi_0(s) | < \infty \qquad \text{a.e. in } S.$$

Let $\{Y_m\}$ be a denumerable dense subset of Y'. Due to (i) and (ii) the decomposability of S and the Radon-Nikodym theorem also imply that for each m a measurable real function $\phi_m(s)$ exists such that

$$\text{(5)} \qquad x_E(y_m) = \int_E \phi_m(s) d\alpha, \qquad E \varepsilon \mathcal{E}_B.$$

Denoting by Y_0 the set of all finite rational linear combinations of the y_m's we then have

$$x_E(y) = \sum_1^n h_j x_E(y_j) = \int_E \sum_1^n h_j \phi_j(s) d\alpha, \qquad E \varepsilon \mathcal{E}_B,$$

whenever $y \varepsilon Y_0$ and $y = \sum_1^n h_j y_j$. Hence for $y \varepsilon Y_0$ and $E \varepsilon \mathcal{E}_B$ the total variation of $x_E(y)$ over E' is $\int_{E'} |\sum_1^n h_j \phi_j(s)| d\alpha$. But $Y_0 \subset Y'$ since Y' is linear· thus the total variation of $x_E(y)$ over E' is not greater than that of $\|y\|\sigma_E$ when $y \varepsilon Y_0$. If y has the form $y = \sum_1^n h_j y_j$ where each h_j is rational, then we have

$$\int_{E'} \left| \sum_1^n h_j \phi_j(s) \right| d\alpha \leqq \|y\| \nu_{E'} = \int_{E'} \|y\| \phi_0(s) d\alpha, \qquad E' \varepsilon \mathcal{E}_B,$$

whence

$$\text{(6)} \qquad \left| \sum_1^n h_j \phi_j(s) \right| \leqq \phi_0(s) \|y\| \qquad \text{a.e. in } S$$

for each such y. The set Y_0 being denumerable, (6) and (4) yield a null set E_0 such that

$$\left| \sum_1^n h_j \phi_j(s) \right| \leqq \phi_0(s) \left\| \sum_1^n h_j y_j \right\| < \infty \tag{7}$$

holds for every $s \varepsilon S - E_0$ and every set $h_1, \cdots, h_n$ of rational numbers. Now let $h_1, \cdots, h_m$ be any real numbers. If we choose rationals h_j^i such that $\lim_i h_j^i = h_j$ for $j = 1, 2, \cdots, m$, (7) implies that

$$\left| \sum_1^m h_j \phi_j(s) \right| = \lim_i \left| \sum_{j=1}^m h_j^i \phi_j(s) \right|, \qquad s \varepsilon S - E_0,$$

and that

$$\left| \sum_{j=1}^m h_j^i \phi_j(s) \right| \leqq \phi_0(s) \left| \sum_{j=1}^m h_j^i y_j \right| < \infty, \qquad s \varepsilon S - E_0,\ i = 1, 2, \cdots.$$

Hence

$$\left| \sum_1^m h_j \phi_j(s) \right| \leqq \phi_0(s) \left\| \sum_1^m h_j y_j \right\| < \infty \tag{8}$$

holds for every $s \varepsilon S - E_0$ and every finite set $h_1, \cdots, h_m$ of real numbers.

In view of (8), a well known theorem on moments [1, p. 57] leads to the existence, for each $s \varepsilon S - E_0$, of an element x_s in the adjoint X of Y' such that

$$\|x_s\| \leqq \phi_0(s), \qquad x_s(y_m) = \phi_m(s), \qquad \text{for } m = 1, 2, \cdots. \tag{9}$$

Over E_0 let x_s be the zero element of X. Due to the Hahn-Banach theorem on the extension of linear functionals, it may be supposed that not only (9) holds but also that

$$\|x_s\| = \sup_y |x_s(y)|, \qquad y \varepsilon Y',\ \|y\| = 1, \tag{10}$$

for every s. This will be used in establishing (2.1.02). To obtain (2.1.01) let y be arbitrary in Y' and choose a subsequence $\{y_{m_i}\}$ in $\{y_m\}$ with $\lim_i y_{m_i} = y$. Then $\lim_i x_s(y_{m_i}) = x_s(y)$ for every s and $\{x_s(y_{m_i})\}$ is a sequence of measurable functions with

$$|x_s(y_{m_i})| \leqq \phi_0(s) \sup_i \|y_{m_i}\|, \qquad s \varepsilon S - E_0,\ i = 1, 2, \cdots.$$

Applying Lebesgue's convergence theorem shows that $x_s(y)$ is summable over every $E \varepsilon \mathcal{E}_B$ and that

$$\int_E x_s(y)d\alpha = \lim_i \int_E x_s(y_{m_i})d\alpha, \qquad E \in \mathcal{E}_B. \tag{11}$$

From (5) and (9) we also have

$$x_E(y) = \lim_i x_E(y_{m_i}) = \lim_i \int_E \phi_{m_i}(s)d\alpha = \lim_i \int_E x_s(y_{m_i})d\alpha \tag{12}$$

for $E \in \mathcal{E}_B$. Relation (2.1.01) now follows by combining (11) and (12). To prove (2.1.02) let $\{z_n\}$ be denumerable and dense in the surface of the unit sphere of Y'. We then have from (10)

$$\|x_s\| = \limsup_n |x_s(z_n)|, \qquad s \in S, \tag{13}$$

so that $\|x_s\|$ is measurable and therefore summable over every $E \in \mathcal{E}_B$. From (iii) and (10) it is also evident that $\sigma_E = \lim\sup_n |x_E(z_n)|$ for every $E \in \mathcal{E}_B$. Relation (2.1.01) then implies that $\sigma_E = \lim\sup_n |\int_E x_s(z_n)d\alpha|$ and hence $\sigma_E \leqq \int_E \lim\sup_n |x_s(z_n)| d\alpha = \int_E \|x_s\| d\alpha$. Thus $\nu_E \leqq \int_E \|x_s\| d\alpha \leqq \int_E \phi_0(s)d\alpha = \nu_E$ by (3) and (9), so that (2.1.02) is true.

Second proof. We first cite the following

LEMMA. (Doob.) *Let Y be a metric space and Y_0 a denumerable set in Y which has the point θ in Y as a point of accumulation. Let $K(y, s)$ be a real function defined for $y \in Y_0$ and $s \in S$ which is measurable in s for each $y \in Y_0$. Suppose further that for any sequence $\{Y_i\}$ of points in Y_0 with* $\lim_i y_i = \theta$ *we have* $\lim K(y_i, s) = 0$ *a.e. in S. Then there is a null set E_0 such that*

$$\lim_{y \to \theta, y \in Y_0} K(y, s) = 0, \qquad s \in S - E_0.$$

In a slightly different form this has been proved by Doob [7, Lemma 2, p. 758], so that we omit the proof here.

If x_E satisfies the assumptions of the theorem, it follows from the decomposability of S and the Radon-Nikodym theorem that there is a real function $K(y, s)$ defined for $y \in Y'$ and $s \in S$ which is measurable in s for each y and is such that

$$x_E(y) = \int_E K(y, s)d\alpha, \qquad y \in Y', E \in \mathcal{E}_B. \tag{14}$$

As in the first proof a measurable non-negative $\phi_0(s)$ exists satisfying (3) and (4). Since (14) implies that over each $E' \in \mathcal{E}_B$ the total variation of $x_E(y)$ is $\int_{E'} |K(y, s)| d\alpha$ and since this total variation is not greater than that of $\|y\|\sigma_E$, we then have

$$\int_{E'} | K(y, s) | \, d\alpha \leqq \|y\| \nu_{E'} = \|y\| \cdot \int_{E'} \phi_0(s) d\alpha, \qquad y \in Y',\ E' \in \mathcal{E}_B,$$

and hence

$$(15) \qquad | K(y, s) | \leqq \|y\| \phi_0(s) \qquad \text{a.e. in } S \text{ for each } y \in Y'.$$

From (14) it follows that

$$(16) \qquad K(ay + by', s) = aK(y, s) + bK(y', s)$$

holds a.e. in S. The set of points in S where (16) fails to hold may of course vary with the real numbers a and b as well as with the vectors y and y'. Let Y_0 consist of all finite linear combinations, with rational coefficients, of the elements of some sequence dense in the span Γ of Y'. Since Y' is linear we may take Y_0 to be in Y'. Due to (16) there is a null set E_0' such that

$$(17) \qquad K(ay + by', s) = aK(y, s) + bK(y', s)$$

holds for y and y' in Y_0, a and b rational, and $s \in S - E_0'$. In view of (4), (15), and the inclusion of Y_0 in Y' we may suppose E_0' so chosen that

$$(18) \qquad | K(y, s) | \leqq \|y\| \phi_0(s) < \infty, \qquad y \in Y_0,\ s \in S - E_0'.$$

Now using (18) together with the lemma of Doob we see that there is a null set E_0 containing E_0' and such that

$$(19) \qquad \lim_{y \to \theta,\, y \in Y_0} K(y, s) = 0, \qquad s \in S - E_0.$$

Thus for each $s \in S - E_0$ there is a $\delta(s) > 0$ such that

$$(20) \qquad | K(y, s) | \leqq 1, \qquad \text{if } y \in Y_0 \text{ and } \|y\| \leqq \delta(s).$$

For each nonzero $y \in Y_0$ choose $\delta_y(s)$ so that $\delta_y(s)/\|y\|$ is rational and $\delta(s)/2 \leqq \delta_y(s) \leqq \delta(s)$. Then from (17) and (20)

$$\left| \frac{\delta_y(s)}{\|y\|} K(y, s) \right| = \left| K\left(\frac{y\delta_y(s)}{\|y\|}, s \right) \right| \leqq 1$$

holds for $s \in S - E_0$ and $y \in Y_0$, so that

$$(21) \qquad | K(y, s) | \leqq \frac{2}{\delta(s)} \|y\|, \qquad s \in S - E_0,\ y \in Y_0.$$

From (17) and (21) we have for $s \in S - E_0$ that

$$(22) \qquad | K(y, s) - K(y', s) | \leqq \frac{2}{\delta(s)} \|y - y'\|, \qquad y \in Y_0,\ y' \in Y_0,$$

which shows that for each such s there is a uniquely defined point x_s in Γ^* with

$$x_s(y) = K(y, s), \qquad y \in Y_0. \tag{23}$$

It may be supposed that the functional x_s over Γ has been extended without increase of norm to the whole of Y and is defined also for $s \in E_0$ by the equation $x_s = \theta_X$. Thus (10) holds. The equalities (14) and (23) show that (2.1.01) is valid for $y \in Y_0$ and the statement (2.1.01) in its entirety follows immediately from this and Lebesgue's convergence theorem since Y_0 is dense in Y' and (18) is true. The conclusion (2.1.02) follows as in the preceding proof since (10), (18), and (23) hold.

Neither conclusion in the above theorem remains true if only (i) and (ii) are assumed. With regard to (2.1.01) it is sufficient to consider the completely additive and absolutely continuous (with respect to α) function defined to the Hilbert space L_2 in Example 9.4 of [31]. This set function is not an integral in the present sense and therefore no function x_s exists satisfying (2.1.01) when $Y' = L_2$. Example 7 of [2] exhibits an integrable function x_s defined to L_2 from $[0, 1]$ for which $\|x_s\|$ is not summable. Here the vector integral $x_E = \int_E x_s d\alpha$ satisfies (i), (ii), and (2.1.01) for $Y' = L_2$ and yet $\|x_s\|$ is not summable.

The following corollary is fundamental for this chapter.

THEOREM 2.1.1. *Let X be the adjoint of another space Y. Suppose x_E is an additive function defined from $\mathcal{E}_B$ to X and that a finite constant K exists such that $\|x_E\| \leqq K\alpha(E)$ for $E \in \mathcal{E}_B$. If Y' is a separable subset of Y, there exists a function x_s defined from S to X and possessing these two properties*:

$$\text{ess. sup. } \|x_s\| \leqq K, \tag{2.1.11}$$

$$x_E(y') = \int_E x_s(y')d\alpha, \qquad E \in \mathcal{E}_B,\ y' \in Y'. \tag{2.1.12}$$

That is, for each such subset Y' in Y there is an essentially bounded function which is Y'-integrable over each $E \in \mathcal{E}_B$ to the value x_E.

A particular case of Theorem 2.1.1 is

THEOREM 2.1.2. *Suppose X is the adjoint Y^* of a separable space Y and that x_E is an additive function defined from $\mathcal{E}_B$ to X and Lipschitzean with constant K. Then there exists a function x_s from S to X such that* (i) ess. sup. $\|x_s\| \leqq K$ *and* (ii) $x_E(y) = \int_E x_s(y)d\alpha$ *for every $y \in Y$ and $E \in \mathcal{E}_B$. Omitting sets of measure zero in S the function x_s is unique.*

That x_s exists satisfying (i) and (ii) is obvious from Theorem 2.1.1. If x_s'

is another such function, pick $\{y_n\}$ dense in Y. From (ii) it follows that $\int_E x_s(y_n)d\alpha = \int_E x_s'(y_n)d\alpha$ or every $E \varepsilon \mathcal{E}_B$ and every n, so that $x_s(y_n) = x_s'(y_n)$ holds a.e. in S for each n. This implies that $x_s(y_n) = x_s'(y_n)$, $n = 1, 2, \cdots$, is true for every s in a set S_1 such that $S - S_1 \varepsilon \mathcal{E}_0$. The sequence $\{y_n\}$ being dense in Y, the conclusion is that $x_s = x_s'$ in X for each $s \varepsilon S_1$, and hence $x_s = x_s'$ a.e. in S.

Theorem 2.1.2 may be restated as follows.

THEOREM 2.1.2′. *From the assumptions in Theorem* 2.1.2 *it follows that there exists an* $x_. \varepsilon \mathfrak{L}_0^\infty(S)[Y^*, Y]$ *such that* ess. sup. $\|x_s\| \leqq K$ *and* x_s *is Y-integrable to* x_E *for each* $E \varepsilon \mathcal{E}_B$. *This function* x_s *is essentially unique.*

Suppose the function $x_.$ given in Theorem 2.1.2′ is almost separably-valued. Since Y is a determining manifold for $X = Y^*$, Theorem 1.1.9 implies that $x_. \varepsilon \mathfrak{A}^\infty(S)[X]$. Thus we can state

THEOREM 2.1.3. *If under the assumptions of Theorem* 2.1.2 *the resulting function* $x_.$ *is almost separably-valued, then* $x_. \varepsilon \mathfrak{A}^\infty(S)[X]$ *with* ess. sup. $\|x_s\| \leqq K$ *and* $x_E = \int_E x_s d\alpha$ *for* $E \varepsilon \mathcal{E}_B$. *Omitting sets of measure zero* $x_.$ *is unique.*

Since Y is separable if its adjoint X is separable, two corollaries result quickly from Theorem 2.1.3.

THEOREM 2.1.4. *If X is a separable adjoint space and* x_E *is an additive function from* $\mathcal{E}_B$ *to X satisfying a Lipschitz condition with constant K, there is an* $x_. \varepsilon \mathfrak{A}^\infty(S)[X]$ *such that* (i) ess. sup. $\|x_s\| \leqq K$ *and* (ii) $x_E = \int_E x_s d\alpha$, $E \varepsilon \mathcal{E}_B$.

THEOREM 2.1.5. *Suppose X is a reflexive space and* x_E *fulfills the assumptions of Theorem* 2.1.4. *Then the conclusions of that theorem hold provided that the values of* x_E *form a separable set in X.*

Theorem 2.1.5 is implied by Theorem 2.1.4. For the functional values of x_E lie together in a separable c.l.m. X' in X. Since X is reflexive so is† X', and hence X' is a separable adjoint space.

B. **Abstract representations of operations to certain adjoint spaces.** The preceding results will now be applied to obtain, in terms of abstract integrals, representations for the general operation sending $L(S)$ into certain adjoint spaces.

THEOREM 2.1.6. *Suppose X is the adjoint Y^* of a separable space Y. If* $x_. \varepsilon \mathfrak{L}^\infty(S)[Y^*, Y]$, *then* $x_. \varepsilon \mathfrak{L}_0^\infty(S)[Y^*, Y]$ *and the Y-integral*

$$U(\phi) = \int_S x_s\phi(s)d\alpha \tag{2.1.6*}$$

† By a theorem of Plessner; see [16, 18].

defines an operation U from $L(S)$ to X. The function $x.$ is essentially bounded and $|U| =$ ess. sup. $\|x_s\|$.

Conversely, if U is an operation from $L(S)$ to X, there is an essentially unique $x.$ in $\mathfrak{L}_0^\infty(S)[Y^*, Y]$ such that $U(\phi)=\int_S x_s\phi(s)d\alpha$, the integral being the Y-integral. The norm of U is $|U| =$ ess. sup. $\|x_s\|$.

The first part of this is included in Theorem 1.1.5. To establish the second half let $x_E = U(\phi_E)$ where ϕ_E is the characteristic function of $E \in \mathcal{E}_B$. Then x_E is a set function additive and Lipschitzean over $\mathcal{E}_B$. From Theorem 2.1.2 it follows that there is an essentially unique $x. \in \mathfrak{L}_0^\infty(S)[Y^*, Y]$ having its Y-integral over E coinciding with x_E for every $E \in \mathcal{E}_B$. But since $x.$ is in $\mathfrak{L}_0^\infty(S)[Y^*, Y]$, by the first part of the theorem the Y-integral $\int_S x_s\phi(s)d\alpha$ defines a second operation $U'(\phi)$ from $L(S)$ to X, with $|U'| =$ ess. sup. $\|x_s\|$. Since U and U' are linear and $U(\phi_E)=U'(\phi_E)$ for each $E \in \mathcal{E}_B$, U and U' are identical over $L(S)$. Therefore $U(\phi)=\int_S x_s\phi(s)d\alpha$, where $x. \in \mathfrak{L}_0^\infty(S)[Y^*, Y]$, and $x.$ is essentially unique and essentially bounded with ess. sup. $\|x_s\| = |U|$.

Under certain circumstances an operation U from $L(S)$ to X may have a stronger representation, in the sense that the Y-integral in (2.1.6*) becomes an integral. For example there is

THEOREM 2.1.7. *If U is an operation from $L(S)$ to the adjoint X of a separable space Y and if the function $x.$ defining U by means of* (2.1.6*) *is almost separably-valued, then $x. \in \mathfrak{A}^\infty(S)[X]$ and $U(\phi)=\int_S x_s\phi(s)d\alpha$. The norm of U is $|U| =$ ess. sup. $\|x_s\|$.*

From Theorem 1.1.7 and the properties of $x.$ it is clear that $x. \in \mathfrak{A}^\infty(S)[X]$ and that the integral $U'(\phi)=\int_S x_s\phi(s)d\alpha$ exists and defines an operation from $L(S)$ to X with $|U'| =$ ess. sup. $\|x_s\|$. On the other hand, $U(\phi)$ is the Y-integral of $x_s\phi(s)$ for each $\phi(.)$. Since Y's being a determining manifold for $X=Y^*$ implies that this Y-integral coincides with the integral $\int_S x_s\phi(s)d\alpha$, we conclude that $U(\phi)=\int_S x_s\phi(s)d\alpha$ for each ϕ, proving the theorem.

Theorems 2.1.6 and 2.1.7. clearly yield

THEOREM 2.1.8. *Let X be the adjoint Y^* of some space Y and suppose X is separable. If $x.$ is any element of $\mathfrak{L}^\infty(S)[Y^*, Y]$, then $x. \in \mathfrak{A}^\infty(S)[X]$, and $U(\phi)=\int_S x_s\phi(s)d\alpha$ is an operation from $L(S)$ to X with $|U| =$ ess. sup. $\|x_s\|$.*

Conversely, given an operation U from $L(S)$ to this space X there is an essentially unique $x.$ such that $x. \in \mathfrak{A}^\infty(S)[X]$ and $U(\phi)=\int_S x_s\phi(s)d\alpha$.

This in turn leads to

THEOREM 2.1.9. *If X has a weakly compact unit sphere, an operation U is defined and separable from $L(S)$ to X if and only if there is an $x. \in \mathfrak{A}^\infty(S)[X]$*

such that $U(\phi)=\int_S x_s\phi(s)d\alpha$. *The norm of* U *is* $|U|=\text{ess. sup.}\,\|x_s\|$, *and* $x.$ *is essentially unique.*

Let X' be the separable span of $U(L(S))$ in X. Since X' is separable and has a weakly compact unit sphere, it is a separable adjoint space [1, p. 199]. By Theorem 2.1.8 there is an essentially unique $x.\ \varepsilon\ \mathfrak{A}^\infty(S)[X']$ such that $U(\phi)=\int_S x_s\phi(s)d\alpha$. Moreover $|U|=\text{ess. sup.}\,\|x_s\|$. These considerations prove the theorem when combined with the corollary to Theorem 1.2.4.

Part 2. Representations of separable operations to $L^q(T)$, $1<q\leqq\infty$

For Euclidean S and T the general operation from $L(S)$ to $L^q(T)$, $1\leqq q\leqq\infty$, has been represented by B-space-valued functions [18, 9, 23, 24] and by kernel integrals [18, 9, 23, 42, 24, 25]. For $q>1$ the representations by means of vector integrals and kernel integrals due to Gelfand [18] and Dunford [9] are here extended to abstract S and T for which either $\mathcal{E}_B$ or $\mathcal{J}_B$ is separable. The case $q=1$ will be considered in Parts 3 and 4.

A. **Operations with range in** $L^q(T)$, $1<q<\infty$. The space $L^q(T)$ having a weakly compact unit sphere when $1<q<\infty$, Theorem 2.1.9 has as a corollary [18, 9]

Theorem 2.2.1. *An operation* U *is defined and separable from* $L(S)$ *to* $L^q(T)$, $1<q<\infty$, *if and only if there exists an* $x.\ \varepsilon\ \mathfrak{A}^\infty(S)[L^q(T)]$ *such that* $U(\phi)=\int_S x_s\phi(s)d\alpha$. *Here* $x.$ *is essentially unique and* $|U|=\text{ess. sup.}\,\|x_s\|$.

A corresponding theorem in terms of kernels is the following [9].

Theorem 2.2.2. *Given a separable operation* U *from* $L(S)$ *to* $L^q(T)$, $1<q<\infty$, *there is a kernel* $K(s, t)$ *such that*

$$U(\phi)=\int_S K(s,t)\phi(s)d\alpha,\qquad \phi\ \varepsilon\ L(S),\tag{2.2.2*}$$

and

(i) *$K(s, t)$ is measurable, that is, $\mathcal{E}\times\mathcal{J}$ measurable, over $S\times T$,*

(ii) $\text{ess. sup.}\,(\int_T|K(s,t)|^q d\beta)^{1/q}\equiv M<\infty$,

(iii) *there exist an* $E_0\ \varepsilon\ \mathcal{E}_0$ *and a separable c.l.m.* $Y\subset L^q(T)$ *such that* $K(s,.)\ \varepsilon\ Y$ *for* $s\notin E_0$.

Conversely, if $K(s, t)$ *satisfies* (i) *and if*

(iv) $K(s,.)\ \varepsilon\ L^q(T)$ *for almost every* s, *and*

(v) $\text{ess. sup.}\,|\int_T K(s,t)\psi'(t)d\beta|<\infty$ *for each* $\psi'\ \varepsilon\ L^{q'}(T)$,

then $K(s, t)$ *satisfies* (i)–(iii) *and the operation* U *of* (2.2.2*) *is defined and separable from* $L(S)$ *to* $L^q(T)$.

In either case the norm of U *is the constant* M *in* (ii).

Any separable operation U from $L(S)$ to $L^q(T)$, $1<q<\infty$, must have a representation $U(\phi)=\int_S x_s\phi(s)d\alpha$ where $x. \ \varepsilon \ \mathfrak{A}^\infty(S)[L^q(T)]$. From Theorem 1.3.5 there exists a kernel $K(s, t)$ satisfying (i), (ii), and (2.2.2*) and such that

$$x_s = K(s, .) \qquad \text{in } L^q(T) \text{ for each } s. \tag{1}$$

Since $x. \ \varepsilon \ \mathfrak{A}^\infty(S)[L^q(T)]$, (1) implies that $K(s, t)$ also satisfies (iii). From Theorem 1.3.5, $|U|=M$. The second half of the theorem is included in Theorem 1.3.7.

For Euclidean S and T the following particular case of Theorem 2.2.2 has already been established by methods depending on the differentiation either of B-space-valued functions [9] or of real functions [25].

Theorem 2.2.3. *If either $\mathcal{E}_B$ or $\mathcal{J}_B$ is separable, it follows that U is an operation from $L(S)$ to $L^q(T)$, $1<q<\infty$, if and only if* (2.2.2*) *holds for some kernel satisfying* (i) *and* (ii) *of Theorem* 2.2.2. *The norm of U is $|U|=M$.*

B. **Operations with range in** $L^\infty(T)$. On recalling that $L^\infty(T)$ is the adjoint of $L(T)$ [1, 30, 10] and that $L(T)$ is separable if $\mathcal{J}_B$ is separable, it is obvious that Theorem 2.1.6 yields

Theorem 2.2.4. *If $\mathcal{J}_B$ is separable, a necessary and sufficient condition that U be an operation defined from $L(S)$ to $L^\infty(T)$ is that there exist an $x. \ \varepsilon \ \mathfrak{L}_0^\infty(S)[L^\infty(T), L(T)]$ such that $U(\phi)=\int_S x_s\phi(s)d\alpha$, the integral being the $L(T)$-integral. The function $x.$ is necessarily essentially bounded and essentially unique, and $|U|=$*ess. sup. $\|x_s\|$.

An analogue to Theorem 2.2.4 in terms of kernels is [42, 18, 24]

Theorem 2.2.5. *Suppose $\mathcal{J}_B$ to be separable. Then U is an operation from $L(S)$ to $L^\infty(T)$ only if there is a $K(s, t)$ having the properties*

(i) *$K(s, t)$ is measurable,*
(ii) *$K(s, t)$ is essentially bounded,*
(iii) *for every ϕ in $L(S)$*

$$U(\phi) = \int_S K(s, t)\phi(s)d\alpha. \tag{2.2.5*}$$

Conversely, if K is a given measurable kernel satisfying the conditions

(iv) ess. sup.$_t$ $|K(s, t)| < \infty$ *for almost every s, and*
(v) ess. sup.$_s$ $|\int_T K(s, t)\psi'(t)d\beta| < \infty$ *for every $\psi' \ \varepsilon \ L(T)$,*

then K satisfies (ii) *and the mapping U given in* (2.2.5*) *exists and is an operation from $L(S)$ to $L^\infty(T)$.*

*The norm of U is $|U|=$*ess. sup. $|K(s, t)|$.

Consider an arbitrary operation $U(\phi)=\mu_\phi$ from $L(S)$ to $L^\infty(T)$ where $\mathcal{J}_B$

is separable. Let $x.$ be the element of $\mathfrak{L}_0^\infty(S)[L^\infty(T), L(T)]$ associated with U according to Theorem 2.2.4. Taking a decomposition $\{T_j\}$ of T, for each j let x_s^j be the "projection" of x_s onto $L^\infty(T_j)$, that is, x_s^j for each s is the essentially bounded F-measurable function over T_j that coincides with x_s everywhere in T_j. Clearly $x_.^j \varepsilon \mathfrak{L}_0^\infty(S)[L^\infty(T_j), L(T_j)]$ for each fixed j, so that $x_s^j(\psi')$ is measurable in s whenever x_s^j operates on an element ψ' of $L(T_j)$. Since $\beta(T_j) < \infty$ the function $x_.^j$ may be considered as defined to $L(T_j)$ where $L(T_j)$ is separable since $\mathcal{J}_B$ is separable. So considered $x_.^j$ is measurable by Theorem 1.1.7. For $L(T_j)$ is separable, and $x_s^j(\psi')$ is measurable for each $\psi' \varepsilon L(T_j)$ and hence for each $\psi' \varepsilon L^\infty(T_j)$; thus $x_.^j$ is separably-valued and weakly measurable as defined to $L(T_j)$. Theorem 1.3.2 implies, then, that a measurable kernel $K_j(s, t)$ exists over $S \times T_j$ such that $x_s^j = K_j(s, .)$ in $L(T_j)$ for each s. Hence for each j and each s we have $K_j(s, t)$ coinciding a.e. in T_j with x_s. Letting $K(s, t) = K_j(s, t)$ for $s \varepsilon S$ and $t \varepsilon T_j$, K is evidently measurable and, for each s, $K(s, t)$ coincides a.e. in T with x_s.

Since $|U| = \text{ess. sup.} \|x_s\|$ we now have $|U| = \text{ess. sup.}_s \text{ ess. sup.}_t |K(s, t)|$ and hence by the Fubini theorem $|U| = \text{ess. sup.} |K(s, t)|$, so that (ii) is true. Moreover, since $\mu_\phi = U(\phi)$ is the $L(T)$-integral of $x_s\phi(s)$ over S and $x_s = K(s, .)$ in $L^\infty(T)$, we can write

$$\int_T \mu_\phi(t)\psi'(t)d\beta = \int_S \phi(s)\left\{\int_T K(s, t)\psi'(t)d\beta\right\}d\alpha, \qquad \psi' \varepsilon L(T),\ \phi \varepsilon L(S);$$

and again applying the Fubini theorem,

$$\int_T \mu_\phi(t)\psi'(t)d\beta = \int_T \psi'(t)\left\{\int_S \phi(s)K(s, t)d\alpha\right\}d\beta, \qquad \psi' \varepsilon L(T),\ \phi \varepsilon L(S).$$

This implies that $\mu_\phi(t) = \int_S \phi(s)K(s, t)d\alpha$ a.e. in T, and hence (iii) is true.

If on the other hand K is a measurable kernel fulfilling (iv) and (v), it is clear that the function $x_s \equiv K(s, .)$ is an element of $\mathfrak{L}^\infty(S)[L^\infty(T), L(T)]$. Hence $x. \varepsilon \mathfrak{L}_0^\infty(S)[L^\infty(T), L(T)]$ since $L(T)$ is separable. Applying Theorem 2.2.4, the $L(T)$-integral $U(\phi) = \mu_\phi = \int_S x_s\phi(s)d\alpha$ defines an operation U to $L^\infty(T)$ having $|U| = \text{ess. sup.} \|x_s\|$. Since K is measurable and $|U| = \text{ess. sup.}_s \text{ ess. sup.}_t |K(s, t)|$, it follows from the Fubini theorem that K is essentially bounded over $S \times T$ and that $|U| = \text{ess. sup.}_{(s,t)} |K(s, t)|$. Hence if $\phi \varepsilon L(S)$ and $\psi' \varepsilon L(T)$ the numerical function $K(s, t)\phi(s)\psi'(t)$ is summable over $S \times T$ and by Fubini's theorem

$$\int_T \psi'(t)\left\{\int_S K(s, t)\phi(s)d\alpha\right\}d\beta = \int_S \phi(s)\left\{\int_T K(s, t)\psi'(t)d\beta\right\}d\alpha.$$

Now from the definition of the $L(T)$-integral we also have

$$\int_T \mu_\phi(t)\psi'(t)d\beta = \int_S \phi(s)\left\{\int_T K(s, t)\psi'(t)d\beta\right\}d\alpha$$

for each $\phi \varepsilon L(S)$ and every $\psi' \varepsilon L(T)$. Thus for each ϕ the equality

$$\int_T \mu_\phi(t)\psi'(t)d\beta = \int_T \psi'(t)\left\{\int_S K(s, t)\phi(s)d\alpha\right\}d\beta$$

holds for every $\psi' \varepsilon L(T)$, which implies that $\mu_\phi = \int_S K(s, t)\phi(s)d\alpha$ in $L^\infty(T)$. This establishes the second half of the theorem, including the fact that $|U| =$ ess. sup. $|K(s, t)|$.

The representation given in Theorem 2.2.5 is a slight variation on a result stated for bounded Euclidean S and T by Gelfand [18]. A proof of Gelfand's theorem has been given by Kantorovitch and Vulich [24] using linear partially ordered spaces and differentiation theory for real functions. Another representation is due to Vulich [42].

As a corollary to Theorem 2.2.5 there is

THEOREM 2.2.6. *When U is separable from $L(S)$ to $L^\infty(T)$ there is a kernel K defined over $S \times T$ and satisfying conditions* (i)–(iii) *of Theorem* 2.2.5. *The norm of U is $|U| =$* ess. sup. $|K(s, t)|$.

Since in $L^\infty(T)$ the span X of $U(L(S))$ is a separable c.l.m., there exists in $\mathcal{J}$ a Borel field $\mathcal{J}'$ such that $\mathcal{J}_B'$ is separable and X is a subset of the set consisting of all elements of $L^\infty(T)$ which are $\mathcal{J}'$-measurable.* Thus X is equivalent to a subset of the adjoint of Y where Y is the B-space of all real functions $\mathcal{J}'$-measurable and β-summable over T. Since U is defined to X and $\mathcal{J}_B'$ is separable, the present theorem follows from the last one.

From Theorem 2.2.6 it is clear that Theorem 2.2.5 holds when $\mathcal{J}_B$ is replaced by $\mathcal{E}_B$.

Theorem 2.2.4 cannot be strengthened to read "there exists an $x.\ \varepsilon\ \mathfrak{A}^\infty(S)[L^\infty(T)]$ such that $U(\phi) = \int_S x_s\phi(s)d\alpha$." Let $S = T = [0, 1]$ and for each s take x_s to be the characteristic function of the interval $[0, s]$. Evidently $x.\ \varepsilon\ \mathfrak{L}_0^\infty(S)[L^\infty(T), L(T)]$, so that the $L(T)$-integral of $x_s\phi(s)$ defines an operation U to $L^\infty(T)$. If there were an $x.'$ in $\mathfrak{A}^\infty(S)[L^\infty(T)]$ such that $U(\phi) = \int_S x_s'\phi(s)d\alpha$, Theorem 2.2.4 would imply that $x_s = x_s'$ a.e. in S, so that x_s would be measurable and therefore almost separably-valued. But from the definition of x_s this is clearly false.

* This is a consequence of the following theorem: *a necessary condition that a subset X of $L^q(T)$, $1 \leqq q \leqq \infty$, be separable is that there exist in $\mathcal{J}$ a Borel field $\mathcal{J}'$ having $\mathcal{J}_B'$ separable and such that every element of X is $\mathcal{J}'$-measurable. If $q < \infty$ this condition is also sufficient.* The proof of this is omitted since the essential features can be found in Theorem 3 of [11].

PART 3. THE REPRESENTATIONS OF OPERATIONS TAKING $L(S)$ INTO $C^*(T)$ OR $L(T)$

The remainder of this chapter is devoted to representing those operations sending $L(S)$ into† $C^*(T)$ or $L(T)$ when T is an interval on the real axis (Part 3) and to characterizing some subclasses of these operations (Part 4). In Part 3 the approach in the $L(T)$ case is that used by Vulich [42] and Gelfand [18] and consists first of choosing a space X composed of numerical functions which include the indefinite integrals of the elements of $L(T)$ and then of representing the general operation U from $L(S)$ to X in the form of a kernel integral. Each operation from $L(S)$ to $L(T)$ is therefore obtained by differentiating a kernel integral sending $L(S)$ into X. As with these two authors our choice of X forces T to be a real interval (finite or infinite) with end points c and d, $-\infty \leqq c < d \leqq \infty$, and requires β to be Lebesgue measure. Both, either, or none of the points c and d may be in T. Due to Theorem 2.1.6 we are able however to avoid one restriction present in the representation theorem as proved by Vulich and Gelfand, namely, that S be a bounded real interval. The present proof permits S to be arbitrary.

Let $BV(T)$ be the class of those numerical functions $\nu(t)$ each of which is of bounded variation over T; that is, $BV(T)$ consists of those $\nu(t)$ defined over T for which

$$\operatorname{var}_t \nu(t) \equiv \underset{\pi}{\text{l.u.b.}} \sum_1^n \left| \nu(t_i) - \nu(t_{i-1}) \right| < \infty$$

holds as π varies over all partitions of the form $\pi = [t_0 < t_1 < \cdots < t_n]$ where $t_i \in T$, $i = 0, 1, \cdots, n$. Each $\nu(.) \in BV(T)$ has the limits $\nu(d-0)$ and $\nu(t+0)$ existing for $c \leqq t < d$; if for a fixed $\nu(.) \in BV(T)$ it is also true that $\nu(c+0) = 0$, $\nu(t+0) = \nu(t)$ for every $t \in T$, and $\nu(d-0) = \nu(d)$ in case $d \in T$, then we say that $\nu(.)$ is in $C^*(T)$. With the norm $\|\nu\| = \rho(\nu) + \operatorname{var}_t \nu(t)$, where $\rho(\nu) = |\nu(c)|$ or $|\nu(c+0)|$ according as $c \in T$ or $c \notin T$, $BV(T)$ is a B-space. In $BV(T)$ the set $C^*(T)$ forms a c.l.m. over which $\|\nu\| = \operatorname{var}_t \nu(t)$, and $C^*(T)$ in turn contains the c.l.m. $AC(T)$ composed of the indefinite Lebesgue integrals of the elements of $L(T)$. Let $C(T)$ denote those numerical functions $\gamma(t)$ which are defined and continuous over T and have the two limits $\gamma(c+0)$ and $\gamma(d-0)$ existing finitely. $C(T)$ is a separable B-space under the norm $\|\gamma\| = \sup_t |\gamma(t)|$. Moreover, for each $\gamma(.) \in C(T)$ and each $\nu(.) \in C^*(T)$ the improper Stieltjes integral

$$\mathrm{S}\int_c^d \gamma(t)d\nu(t) = \lim_{c',d'} \mathrm{S}\int_{c'}^{d'} \gamma(t)d\nu(t), \qquad c' \to c+0,\ d' \to d-0,$$

† The definition of $C^*(T)$ is given in the next paragraph.

exists and defines over $C(T)$ a linear functional having $\text{var}_t\, \nu(t)$ for its norm. Conversely, given a linear functional $f(\gamma)$ over $C(T)$, exactly one element $\nu(.)$ exists in $C^*(T)$ such that $f(\gamma) = \mathrm{S}\int_c^d \gamma(t) d\nu(t)$ for every γ.

Thus the c.l.m. $C^*(T)$ is equivalent to the adjoint of the separable space $C(T)$ and $C^*(T)$ contains the indefinite integrals of the elements of $L(T)$. Taking $C^*(T)$ as the space X mentioned above† we can apply Theorem 2.1.6 to obtain for the generic operation U sending $L(S)$ into $C^*(T)$ a kernel representation $U(\phi) = \int_S K(s, t)\phi(s) d\alpha$ where K has certain detailed properties (Theorem 2.3.1). Since $C^*(T) \supset AC(T)$, this leads immediately to the result of Vulich and Gelfand, namely, that U' is an operation from $L(S)$ to $L(T)$ if and only if

$$U'(\phi) = \frac{d}{dt} \int_S K(s, t)\phi(s) d\alpha$$

where $K(s, t)$ belongs to a specified class of kernels (Theorem 2.3.9).

We thus suppose T to be a fixed linear interval and $BV(T)$, $C^*(T)$, and $C(T)$ to be the spaces described above. The set of operations mapping $L(S)$ into $C^*(T)$ will be denoted by $\mathfrak{U}$. Finally, it is to be noted that for each $t \in T$ the functional $f_t(\nu) \equiv \nu(t)$ is linear over $C^*(T)$ and $\|f_t\| = 1$.

A. **Operations to** $C^*(T)$. The chief theorem of this part is

THEOREM 2.3.1. *Any operation $U(\phi) = x_\phi$ from $L(S)$ to $C^*(T)$, that is, any $U \in \mathfrak{U}$, has a representation of the form*

$$U(\phi) = x_\phi = \int_S K(s, t)\phi(s) d\alpha, \qquad \phi \in L(S), \tag{2.3.1*}$$

where $K(s, t)$ is a real kernel defined over $S \times T$ and having the following properties:

(2.3.11) *$K(s, .) \in C^*(T)$ for each $s \in S$,*

(2.3.12) *$K(., t) \in L^\infty(S)$ for each $t \in T$,*

(2.3.13) *$K(s, t)$ is measurable and* ess. sup. $|K(s, t)| < \infty$,

(2.3.14) ess. sup.$_s$ $[\text{var}_t\, K(s, t)] \equiv M < \infty$,

(2.3.15) *the two iterated integrals $\mathrm{S}\int_c^d \gamma(t) d\{\int_S K(s, t)\phi(s) d\alpha\}$ and $\int_S \{\mathrm{S}\int_c^d \gamma(t) d_t K(s, t)\}\phi(s) d\alpha$ exist and are equal whenever $\gamma \in C(T)$ and $\phi \in L(S)$.*

Conversely, if $K(s, t)$ satisfies the conditions

(2.3.16) *$K(s, .) \in C^*(T)$ for almost every s,*

(2.3.17) *$K(s, t)$ is measurable in s for every t in a dense subset of T,*

(2.3.18) ess. sup.$_s$ $|\mathrm{S}\int_c^d \gamma(t) d_t K(s, t)| < \infty$ *for each $\gamma(.) \in C(T)$,*

† In [41] and [18] $BV(T)$ is chosen as the space X. For a kernel representation of the general operation from $L(S)$ to $BV(T)$ see Kantorovitch [23], Vulich [42], and Gelfand [18].

then K has the additional properties (2.3.12)–(2.3.15) *and the operation U of* (2.3.1*) *is defined and linear from $L(S)$ to $C^*(T)$, that is, $U \varepsilon \mathfrak{U}$.*

The norm of U in either case is the constant M of (2.3.14).

The proof will consist of a sequence of four theorems each of which will also be useful in later pages. The first of these is an obvious consequence of Theorem 2.1.6.

THEOREM 2.3.2. *A mapping U is an operation $U(\phi)=\nu_\phi$ defined from $L(S)$ to $C^*(T)$ if and only if there is a $\nu.\ \varepsilon\ \mathfrak{L}^\infty(S)[C^*(T), C(T)] = \mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$ such that, in $C^*(T)$, ν_ϕ is the $C(T)$-integral $\int_S \nu_s \phi(s) d\alpha$ for each $\phi \varepsilon L(S)$, that is,*†

$$(2.3.2^*) \qquad \nu_\phi \cdot \gamma = \int_S \nu_s \cdot \gamma \phi_s d\alpha, \qquad \gamma \varepsilon C(T),\ \phi \varepsilon L(S).$$

This function $\nu.$ is essentially unique and ess. sup. $\|\nu_s\| = |U|$.

The second theorem is important.

THEOREM 2.3.3. *Suppose that $K(s, t)$ is a numerical kernel given over $S \times T$ and that there exist an $E_0 \varepsilon \mathcal{E}_0$ and a $\nu.\ \varepsilon\ \mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$ such that*

$$(2.3.31) \qquad K(s, .) = \nu_s \text{ in } C^*(T) \qquad \text{for each } s \varepsilon S - E_0.$$

Let U be the element of $\mathfrak{U}$ defined by the $C(T)$-integral $U(\phi)=\nu_\phi=\int_S \nu_s \phi(s) d\alpha$. Then K has the properties (2.3.12)–(2.3.16) *and $U(\phi)=\int_S K(s, t)\phi(s) d\alpha$ for each $\phi \varepsilon L(S)$. The norm of U is the constant M of* (2.3.14).

In the proof of this the following lemma will be needed.

LEMMA. *If $t' \varepsilon T$ is fixed and $f_{t'}(\nu)$ is the linear functional over $C^*(T)$ defined by $f_{t'}(\nu) \equiv \nu(t')$, then there exists in $C(T)$ a sequence $\{\gamma_n\}$ such that $\|\gamma_n\| \leqq 1$ and $f_{t'}(\nu) = \lim_n \mathrm{S}\int_c^d \gamma_n(t) d\nu(t)$ for every $\nu \varepsilon C^*(T)$. The sequence $\{\gamma_n\}$ is independent of ν and depends only on t'.*

If $t'=c$ or $t'=d$ the respective sequences $\gamma_n(t)\equiv 0$ or $\gamma_n(t)\equiv 1$ will serve. In case $c<t'<d$ let $\{h_n\}$ be positive constants such that $T \supset \{t'+h_n\}$ and $\lim h_n=0$. For each n define $\gamma_n(t)$ to have the value 1 for $c<(=)\ t \leqq t'$, the value 0 for $t'+h_n \leqq t < (=)\ d$, and to be linear between t' and $t'+h_n$. Clearly $\|\gamma_n\| \leqq 1$ and $\mathrm{S}\int_c^d \gamma_n(t) d\nu(t) = \mathrm{S}\int_c^d \gamma_n(t) df_t(\nu) = f_{t'}(\nu) + \mathrm{S}\int_{t'}^{t'+h_n} \gamma_n(t) df_t(\nu)$ for arbitrary $\nu \varepsilon C^*(T)$; thus it is sufficient to prove that $0=\lim_n \mathrm{S}\int_{t'}^{t'+h_n} \gamma_n(t) df_t(\nu)$. There is no loss of generality in supposing the element $\nu(.)$ of $C^*(T)$ to be monotone non-decreasing as a function of t. For a fixed ν of this sort we can then write, due to h_n's being positive,

† For each $\nu \varepsilon C^*(T)$ and each $\gamma \varepsilon C(T)$ the symbol $\nu \cdot \gamma$ represents the value taken at γ by the linear functional defined over $C(T)$ by ν, that is, $\nu \cdot \gamma$ is the value of $\mathrm{S}\int_c^d \gamma(t) d\nu(t)$.

$$\left| \mathrm{S}\int_{t'}^{t'+h_n} \gamma_n(t)df_t(\nu) \right| \leqq \max_t \left| \gamma_n(t) \right| \cdot \operatorname*{var}_t [f_t(\nu); t', t' + h_n]$$
$$= f_{t'+h_n}(\nu) - f_{t'}(\nu),$$

and hence $\lim_n \int_{t'}^{t'+h_n}\gamma_n(t)df_t(\nu) = 0$, since $\nu(t') = \nu(t'+0)$. This establishes the lemma.

In Theorem 2.3.3 it is obvious, in view of (2.3.31), that $K(s, t)$ has property (2.3.16). Moreover, (2.3.31) and Theorem 2.3.2 imply that the $C(T)$-integral $\int_S \nu_s \phi(s)d\alpha = \nu_\phi$ defines an operation $U(\phi) = \nu_\phi$ from $L(S)$ to $C^*(T)$ with $|U| =$ ess. sup. $\|\nu_s\| =$ ess. sup.$_s$ [var$_t$ $K(s, t)$]; thus $K(s, t)$ satisfies (2.3.14). It is also evident from (2.3.31) and (2.3.2*) that

$$\tag{1} \begin{aligned} \mathrm{S}\int_c^d \gamma(t)d_t f_t(\nu_\phi) &= \nu_\phi \cdot \gamma = \int_S \nu_s \cdot \gamma\phi(s)d\alpha \\ &= \int_S \left\{ \mathrm{S}\int_c^d \gamma(t)d_t K(s, t) \right\} \phi(s)d\alpha \end{aligned}$$

holds for every $\gamma \in C(T)$ and every $\phi \in L(S)$. To establish (2.3.12) fix $t' \in T$. According to the lemma there is in $C(T)$ a sequence $\{\gamma_n\}$ independent of s and such that $f_{t'}(\nu_s) = \lim_n \nu_s \cdot \gamma_n$ whenever $\nu_s \in C^*(T)$. Since (2.3.31) implies that for almost every s we have both $\nu_s \in C^*(T)$ and $K(s, t') = f_{t'}(\nu_s)$, the conclusion is that $\lim_n \nu_s \cdot \gamma_n = K(s, t')$ a.e. in S. Hence $K(s, t')$ is measurable, since by assumption $\nu_s \cdot \gamma_n$ is measurable for each n. Moreover,

$$\tag{2} \operatorname*{ess.\,sup.}_s \left| K(s, t') \right| \leqq M, \qquad t' \in T,$$

because ess. sup. $|\nu_s \cdot \gamma_n| \leqq \|\gamma_n\|$ ess. sup. $\|\nu_s\| \leqq M$. Thus (2.3.12) is true.

To see that $K(s, t)$ is measurable over $S \times T$, for each m let $t_{m,j}$, $j = \cdots, -1, 0, 1, \cdots$, be points in T such that $0 < t_{m,j} - t_{m,j-1} < 1/2^m$ and $\lim_{j\to-\infty} t_{m,j} = c$, $\lim_{j\to\infty} t_{m,j} = d$. Define $K_m(s, t) = K(s, t_{m,j})$ when $t_{m,j-1} \leqq t < t_{m,j}$. From (2.3.12), $K(s, t_{m,j})$ is measurable in s, so that $K_m(s, t)$ is measurable in $S \times (c, d)$. Since $K(s, t+0) = K(s, t)$ for $t < d$ and $s \in S - E_0$, it follows that $K(s, t) = \lim_m K_m(s, t)$ for each $s \in S - E_0$ and each t not of the form $t = t_{k,j}$, c, or d. Thus $K(s, t)$ is a.e. in S the limit of a sequence of measurable functions and is therefore measurable. The inequality ess. sup. $|K(s, t)| \leqq M$ now results from (2) and the Fubini theorem. This vindicates (2.3.13).

In showing for each ϕ that $U(\phi) = \nu_\phi = \int_S K(s, t)\phi(s)d\alpha$, that is, that $f_t(\nu_\phi) = \int_S K(s, t)\phi(s)d\alpha$ for all t, use is again made of the lemma. Fixing t, sequence $\{\gamma_n\}$ exists in $C(T)$ independently of ϕ and such that $f_t(\nu_\phi) = \lim_n \nu_\phi \cdot \gamma_n$, where $\|\gamma_n\| \leqq 1$. Applying (1) this becomes

$$f_t(\nu_\phi) = \lim_n \int_S \nu_s \cdot \gamma_n \phi(s) d\alpha, \qquad \phi \in L(S). \tag{3}$$

On the other hand the lemma also yields the information that $f_t(\nu_s)\phi(s) = \lim_n \nu_s \cdot \gamma_n \phi(s)$ for almost all s. Since $f_t(\nu_s)\phi(s)$ equals $K(s, t)\phi(s)$ for all $s \in S - E_0$ by (2.3.31), it follows that $K(s, t)\phi(s) = \lim_n \nu_s \cdot \gamma_n \phi(s)$ a.e. in S. From the inequality $|\nu_s \cdot \gamma_n \phi(s)| \leqq M\|\gamma_n\| |\phi(s)| \leqq M|\phi(s)|$ and Lebesgue's convergence theorem, $K(s, t)\phi(s)$ is summable in s and $\int_S K(s, t)\phi(s) d\alpha = \lim_n \int_S \nu_s \cdot \gamma_n \phi(s) d\alpha$. From (3) we can now write

$$f_t(\nu_\phi) = \int_S K(s, t)\phi(s) d\alpha, \qquad t \in T, \ \phi \in L(S), \tag{4}$$

so that $U(\phi) = \int_S K(s, t)\phi(s) d\alpha$ where $|U| =$ ess. sup. $\|x_s\| = M$. The proof is complete on noting that property (2.3.15) results immediately from (1) and (4).

THEOREM 2.3.4. *Given* $\nu_. \in \mathfrak{L}^\infty(S)[C^*(T), C(T)]$, *a kernel* $K(s, t)$ *exists over* $S \times T$ *such that* (2.3.11) *is satisfied and* (2.3.31) *is true for some* $E_0 \in \mathcal{E}_0$. *Thus all the conclusions of Theorem* 2.3.3 *hold for the kernel* $K(s, t)$ *and the function* $\nu_.$.

Set $E_0 = S[\nu_s \notin C^*(T)]$ and let $K(s, t) = f_t(\nu_s)$ for $s \in S - E_0$ and $t \in T$, $K(s, t) \equiv 0$ for $s \in E_0$. Obviously (2.3.11) is fulfilled, and since $E_0 \in \mathcal{E}_0$ and $\mathfrak{L}^\infty(S)[C^*(T), C(T)] = \mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$ it follows that (2.3.31) holds.

The last of the four theorems needed is

THEOREM 2.3.5. *Let* $K(s, t)$ *over* $S \times T$ *satisfy the conditions* (2.3.17) *and* (2.3.18), *and let* $E_0 \in \mathcal{E}_0$ *be such that* $K(s, .) \in C^*(T)$ *for* $s \in S - E_0$. *Then the function* $\nu_s \equiv K(s, .)$, $s \in S - E_0$, *is in* $\mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$ *and all the conclusions of Theorem* 2.3.3 *are true.*

Since $\mathfrak{L}_0^\infty(S)[C^*(T), C(T)] = \mathfrak{L}^\infty(S)[C^*(T), C(T)]$, it is sufficient to show that $\nu_s \cdot \gamma \in L^\infty(S)$ for each $\gamma \in C(T)$. For each $s \in S - E_0$, $E_0 \in \mathcal{E}_0$, we have $\nu_s \cdot \gamma = \mathrm{S}\int_c^d \gamma(t) dK(s, t)$, so that ess. sup. $|\nu_s \cdot \gamma| < \infty$ from (2.3.18). And from (2.3.17) it follows that $\nu_s \cdot \gamma$ is measurable in s. For points $\{t_{n,i}\}$, $n = 1, 2, \cdots$, $i = \cdots, -1, 0, 1, \cdots$, can be so chosen in T that $0 < t_{n,i} - t_{n,i-1} < 1/2^n$, $\lim_{i \to -\infty} t_{n,i} = c$, $\lim_{i \to \infty} t_{n,i} = d$, and $K(s, t_{n,i})$ is measurable in s. Setting

$$p_n(\gamma, s) = \sum_{i=-\infty}^{\infty} \gamma(t_{n,i})[K(s, t_{n,i}) - K(s, t_{n,i-1})]$$

it is clear for each $\gamma \in C(T)$ and each n that $p_n(\gamma, s)$ exists for $s \in S - E_0$ and is measurable. Moreover,

$$\nu_s \cdot \gamma = \mathrm{S}\int_c^d \gamma(t) dK(s, t) = \lim_n p_n(\gamma, s)$$

for $s \in S-E_0$. Hence for each γ, $\nu_s \cdot \gamma$ is the limit a.e. of measurable functions, and so is itself measurable. This ends the proof.

The demonstration of Theorem 2.3.1 can now be easily constructed. If U is in $\mathfrak{U}$, there is, by Theorem 2.3.2, a $\nu_. \in \mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$ such that $U(\phi)$ is the $C(T)$-integral $\int_S \nu_s \phi(s) d\alpha$ for each ϕ. From Theorem 2.3.4 a kernel $K(s, t)$ exists over $S \times T$ having the properties (2.3.11)–(2.3.15) and such that $U(\phi) = \int_S K(s, t)\phi(s) d\alpha$, where $|U|$ is the constant M of (2.3.14). If, on the other hand, $K(s, t)$ is given satisfying (2.3.16)–(2.3.18), then by Theorem 2.3.5 $U(\phi) = \int_S K(s, t)\phi(s) d\alpha$ is an operation from $L(S)$ to $C^*(T)$ with $|U| = M$.

For a given kernel $H(s, t)$ varying sets of conditions can be proved to be sufficient in order that the operation $U(\phi) = \int_S H(s, t)\phi(s) d\alpha$ should be a point in $\mathfrak{U}$. The following substitution theorem may therefore be of some use in that it asserts that each measurable kernel of this sort is equivalent, as far as measure is concerned, to a second kernel which satisfies (2.3.11)–(2.3.15). Moreover this second kernel defines the same operation, so that it can serve as a replacement for $H(s, t)$.

THEOREM 2.3.6. *Given a measurable kernel $H(s, t)$ with the property that $U(\phi) = \int_S H(s, t)\phi(s) d\alpha$ is in $\mathfrak{U}$, there exists a second kernel $K(s, t)$ such that*

(2.3.61) $U(\phi) = \int_S K(s, t)\phi(s) d\alpha, \phi \in L(S)$,

(2.3.62) *$K(s, t)$ satisfies* (2.3.11)–(2.3.15),

(2.3.63) *the function $\nu_s \equiv K(s, .)$ is in $\mathfrak{L}^\infty(S)[C^*(T), C(T)]$,*

(2.3.64) *$t \in T$ implies $K(s, t) = H(s, t)$ a.e. in S,*

(2.3.65) *$K(s, t) = H(s, t)$ a.e. in $S \times T$,*

(2.3.66) *for almost all s the equality $K(s, t) = H(s, t)$ holds a.e. in T.*

According to Theorem 2.3.1 there is a $K(s, t)$ such that (2.3.61) and (2.3.62) are true. Hence for each t the two integrals $\int_S H(s, t)\phi(s) d\alpha$ and $\int_S K(s, t)\phi(s) d\alpha$ coincide for every $\phi \in L(S)$, which implies that, for each t, $K(s, t) = H(s, t)$ a.e. in S; thus (2.3.64) holds. (2.3.65) now follows from the Fubini theorem and the measurability of the two kernels, and (2.3.66) is a consequence of (2.3.65). Finally, since (2.3.11)–(2.3.15) imply (2.3.16)–(2.3.18) it is seen that (2.3.63) follows from (2.3.62) and Theorem 2.3.5.

A given kernel $H(s, t)$ may define a linear operation $\int_S H(s, t)\phi(s) d\alpha$ from $L(S)$ to $C^*(T)$ and yet possess very few desirable properties beyond the obvious ones that $H(., t)$ is in $L^\infty(S)$ for each t and that $K(s, t)$ exists satisfying (2.3.11)–(2.3.15) and having $K(s, t) = H(s, t)$ a.e. in S for each t. As an example, in the unit square let $H(s, t)$ be the characteristic function of Sierpiński's non-measurable set that has at most two points in common with any straight line [36]. Here $\int_S H(s, t)\phi(s) d\alpha$ sends each ϕ into the identically zero function; yet $H(s, t)$ is non-measurable, $|U| \neq$ ess. $\sup._s$ [$\text{var}_t\, H(s, t)$], and

(2.3.16) fails to be true. The final two theorems of this part show that with an additional assumption it can be concluded that $H(s, t)$ does have properties (2.3.12)–(2.3.16).

THEOREM 2.3.7. *Suppose that $H(s, t)$ is measurable, that the mapping $U(\phi) = \int_S H(s, t)\phi(s)d\alpha$ is an element of $\mathfrak{U}$, and that $H(s, t)$ is for almost every s everywhere continuous on the right in t, Then $H(s, t)$ satisfies* (2.3.12)–(2.3.16) *and $\nu_s \equiv H(s, .)$ is in $\mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$.*

If $K(s, t)$ is the kernel given in the conclusion of Theorem 2.3.6 , the equality $H(s, t) = K(s, t)$ holds a.e. in T for almost every s. Since for almost every s it is also true that $H(s, t)$ is continuous on the right over T, it is evident there exists an $E_0 \in \mathcal{E}_0$ such that $s \in S - E_0$ implies $H(s, t) = K(s, t)$ everywhere in T; for two functions of t both continuous on the right and coinciding over a dense set are necessarily identical. Thus $H(s, .)$ is, for almost every s, the element $K(s, .)$ of $C^*(T)$. Since $K(s, .) \in \mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$ by Theorem 2.3.5, it follows from Theorem 2.3.3 that $H(s, t)$ satisfies (2.3.12)–(2.3.16). From this fact and from Theorem 2.3.5 we conclude also that $\nu_s \equiv H(s, .)$ is in $\mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$.

THEOREM 2.3.8. *Suppose $H(s, .)$ is essentially defined to $C^*(T)$ and that $U(\phi) = \int_S H(s, t)\phi(s)d\alpha$ is in $\mathfrak{U}$. Then $\nu_s \equiv H(s, .)$ is in*

$$\mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$$

and all the conclusions of Theorem 2.3.3 *hold. In particular $H(s, t)$ satisfies* (2.3.12)–(2.3.16) *and hence is measurable.*

It is evident that $H(s, t)$ satisfies (2.3.16) and (2.3.17). In view of Theorem 2.3.5 there is need only of showing that ess. $\sup_{.s} |\mathrm{S}\int_c^d \gamma(t)dK(s, t)| < \infty$ for each $\gamma \in C(T)$. This may be proved as follows. The function $h_t = H(., t)$ is defined from T to $L^\infty(S)$ and $h_t \cdot \phi = \int_S H(s, t)\phi(s)d\alpha$ is of bounded variation in t for each $\phi \in L(S)$. Hence from [10] it is seen that h_t is an abstract function having bounded variation in the sense of that paper, and that the abstract integral $V(\gamma) = \mathrm{S}\int_c^d \gamma(t) \cdot d_t h_t$ exists as an element of $L^\infty(S)$ for each $\gamma \in C(T)$. Fix γ and in T choose points $\{t_{n,i}\}$, $n = 1, 2, \cdots, i = \cdots, -1, 0, 1, \cdots$, satisfying the conditions $0 < t_{n,i} - t_{n,i-1} < 1/2^n$ and $\lim_{i\to-\infty} t_{n,i} = c$, $\lim_{i\to\infty} t_{n,i} = d$. For each s such that $H(s, .) \in C^*(T)$ the function $\mu_n(s) = \sum_i \gamma(t_{n,i})[H(s, t_{n,i}) - H(s, t_{n,i-1})]$ is defined and $\lim_n \mu_n(s) = \mathrm{S}\int_c^d \gamma(t)d_t H(s, t)$. On the other hand from the definition of $\mathrm{S}\int_c^d \gamma(t)d_t h_t$ we have $V(\gamma) = \lim_n \mu_n(.)$ in $L^\infty(S)$, so that $\{\mu_n(s)\}$ converges a.e. in S to $\mu(s)$ where $\mu(.) = V(\gamma)$. Since $H(s, .) \in C^*(T)$ for almost all s, we now have $\mathrm{S}\int_c^d \gamma(t)dH(s, t) = \mu(s)$ holding a.e. in S, and hence ess. sup. $|\mathrm{S}\int_c^d \gamma(t)dH(s, t)| < \infty$. This establishes Theorem 2.3.8.

As an immediate corollary we see that if $H(s, .)$ and $K(s, .)$ are essentially defined to $C^*(T)$ and define the same operation $U \varepsilon \mathfrak{U}$ then $H(s, t) = K(s, t)$, $t \varepsilon T$, $s \varepsilon S - E_0$ for some $E_0 \varepsilon \mathcal{E}_0$.

B. **Operations from** $L(S)$ **to** $L(T)$. From Theorem 2.3.1 the reader can easily infer [42, 18]

THEOREM 2.3.9. *Any operation U' from $L(S)$ to $L(T)$ has a representation of the form*

$$U'(\phi) = \psi_\phi = \frac{d}{dt}\int_S K(s, t)\phi(s)d\alpha, \qquad \phi \varepsilon L(S), \tag{2.3.9*}$$

where $\int_S K(s, t)\phi(s)d\alpha \varepsilon AC(T)$ for each ϕ and $K(s, t)$ has properties (2.3.11)–(2.3.15). *The norm of U' is the constant M of* (2.3.14).

Conversely, if $K(s, t)$ satisfies (2.3.16)–(2.3.18), *the operation U' of* (2.3.9*) *is defined from $L(S)$ to $L(T)$ and $K(s, t)$ has properties* (2.3.12)–(2.3.15). *Here $|U'| \leqq M$, with equality holding if $\int_E K(s, t)d\alpha \varepsilon AC(T)$ for each $E \varepsilon \mathcal{E}_B$.*

PART 4. SUBCLASSES OF THOSE OPERATIONS HAVING THEIR RANGE IN $C^*(T)$ OR $L(T)$

In the preceding it was seen that any operation U' from $L(S)$ to $L(T)$ has a representation

$$U'(\phi) = \frac{d}{dt}\int_S H(s, t)\phi(s)d\alpha \tag{2.4.01}$$

where H is measurable and $H(s, .)$ is defined to $BV(T)$ (to $C^*(T)$ actually). Since some operations mapping $L(S)$ into $L(T)$ cannot have the form

$$U'(\phi) = \int K'(s, t)\phi(s)d\alpha \tag{2.4.02}$$

where K' is measurable, the question naturally arises as to when a given U' does have the more convenient representation (2.4.02). In this part an answer is given in the following terms. U' can be written in the form (2.4.02) with K' measurable and ess. $\sup._s \int_T |K'(s, t)| d\beta < \infty$ if and only if among the representations (2.4.01) of U' there is one with H measurable and $H(s, .)$ essentially defined to some separable c.l.m. in $BV(T)$. This result is obtained by establishing in terms of kernels a necessary and sufficient condition that a given $U \varepsilon \mathfrak{U}$ be representable as $U(\phi) = \int_S \nu_s \phi(s)d\alpha$ where $\nu. \varepsilon \mathfrak{A}^\infty(S)[C^*(T)]$. In the third section characterizations by means of kernels are given for the subclass in $\mathfrak{U}$ composed of those operators defined by elements of $\mathfrak{A}^\infty(S)[AC(T)]$.

A. **Operations to $C^*(T)$ given by almost separably-valued kernels.** We begin with

THEOREM 2.4.1. *Suppose $U \epsilon \mathfrak{U}$ has a representation $U(\phi)=\int_S \nu_s \phi(s) d\alpha$ where $\nu. \epsilon \mathfrak{A}^\infty(S)[C^*(T)]$. In $C^*(T)$ let Y be the separable span of $U(L(S))$. Then*

(i) *$\nu.$ is essentially defined to Y,*

(ii) *U can be written as*

$$U(\phi) = \int_S K(s, t)\phi(s) d\alpha$$

where $K(s, t)$ has properties (2.3.11)–(2.3.15) *and $K(s, .)=\nu_s$ for almost every s,*

(iii) *$K(s, .)$ is essentially defined to Y and $K(s, .)$ is almost separably-valued in $C^*(T)$. The norm of U is $|U| =$ess. sup. $\|\nu_s\| =$ess. sup.$_s$ $[\text{var}_t K(s, t)]$.*

From the corollary to Theorem 1.2.4 the span Y is separable in $C^*(T)$ and by Theorem 1.2.9 $\nu.$ is essentially defined to Y. The remaining conclusions can now be derived from Theorem 2.3.4.

In the direction reverse to Theorem 2.4.1 there is the more difficult

THEOREM 2.4.2. *Let $U(\phi)=\int_S H(s, t)\phi(s)d\alpha$ be an operation from $L(S)$ to $C^*(T)$ where H is measurable and $H(s, .)$ is essentially defined to a separable c.l.m. X in $BV(T)$. Then*

(i) *$H(s, .)$ is essentially defined to the separable c.l.m. $Y=C^*(T)\cdot X$ in $C^*(T)$,*

(ii) *$H(s, t)$ has properties* (2.3.12)–(2.3.16),

(iii) *U has a representation $U(\phi)=\int_S \nu_s \phi(s) d\alpha$ where $\nu. \epsilon \mathfrak{A}^\infty(S)[Y]$ and $H(s, .)=\nu_s$ in $C^*(T)$ for almost every s, and*

(iv) *the range of U is in Y and $|U| =$ess. sup. $\|\nu_s\| =$ess. sup.$_s$ $[\text{var}_t H(s,t)]$.*

Let $K(s, t)$ be a kernel representing U according to Theorem 2.3.6 so that for almost every s we have $K(s, t)=H(s, t)$ a.e. in T. Since $H(s, .)$ is essentially defined to X, there is then a null set E_0' such that for each $s \epsilon S-E_0'$

$$H(s, t) = K(s, t) \text{ a.e. in } T, \tag{1}$$

$$H(s, .) \epsilon X \subset BV(T). \tag{2}$$

The subset X being separable, a denumerable set $\{s_n\}$ exists in $S-E_0'$ with the property that for each $s \notin E_0'$ there is a subsequence $\{s_{m_i}\}$ such that $\lim_{i\to\infty} \|H(s, .)-H(s_{m_i}, .)\| =0$, the norm being taken in $BV(T)$. This implies that

$$\lim_{i\to\infty} H(s_{m_i}, t) = H(s, t) \qquad \text{uniformly in } t. \tag{3}$$

Since $H(s_m, t)$ is of bounded variation in t, the set of points in T for which the equality $H(s_m, t)=H(s_m, t+0)$ holds for every m includes all of T except an at most denumerable set. Let $\{t_n\}$ be this exceptional set plus any end points of T that may be in T. From (2.3.64) there is for each n a null set E_n such that $H(s, t_n)=K(s, t_n)$ for $s \notin E_n$. Let $E_0=E_0'+\sum E_n$, so that E_0 is null. For each $s \in S-E_0$ we now have not only (1)–(3) holding but also

$$(4) \qquad H(s, t_n) = K(s, t_n), \qquad n = 1, 2, \cdots .$$

From (3) it is seen that if $s \notin E_0$ and $H(s, t^*+0) \neq H(s, t^*)$ for some t^* then $H(s_{m_i}, t^*+0) \neq H(s_{m_i}, t^*)$ for at least one i. This of course means that $t^*=t_n$ for at least one n. Hence we can conclude that for $s \notin E_0$

$$(5) \qquad H(s, t+0) = H(s, t), \qquad t \notin \{t_n\}.$$

For each fixed $s \in S-E_0$ it now follows that $H(s, t)=K(s, t)$ for all t. For (1) implies that $H(s, t)=K(s, t)$ over a dense set in T so that

$$(6) \qquad H(s, t+0) = K(s, t+0) \qquad \text{for every } t.$$

From (4), (5), (6) and the right continuity of $K(s, t)$ everywhere in T it is then seen that

$$K(s, t) = K(s, t+0) \doteq H(s, t+0) = H(s, t)$$

holds for all t. Since $s \in S-E_0$ implies $K(s, .) \in C^*(T)$ and $H(s, .) \in X$, $H(s, .)$ is essentially defined to Y. This establishes (i). Conclusion (ii) follows from (i) and Theorem 2.3.8. To obtain (iii) we set $\nu_s=K(s, .)$ and note first that $\nu_s=H(s, .)$ for almost all s. From Theorem 2.3.5 and the equality $\nu_s=K(s, .)$ it follows that $\nu. \in \mathfrak{L}_0^\infty(S)[C^*(T), C(T)]$. Since ν_s coincides a.e. over S with the almost separably-valued function $H(s, .)$, Theorem 1.1.9 then implies that $\nu. \in \mathfrak{A}^\infty(S)[C^*(T)]$ and Theorem 2.3.3 asserts that $\int_S H(s, t)\phi(s)d\alpha=\int_S \nu_s\phi(s)d\alpha$ in $C^*(T)$. Thus (iii) is proved. It is a result of (iii) and Theorem 1.2.4 that the range of U is in Y. The theorem is established.

With each $U \in \mathfrak{U}$ there are associated two other operations, $U'(\phi) = d\nu_\phi(t)/dt$ with range in $L(T)$ and

$$V(\phi) = \int_c^\tau \left[\frac{d}{dt}\nu_\phi(t)\right] dt$$

with range in $AC(T)$. The first of these can always be represented in the form (2.3.9*) but not always can it be written as $U'(\phi)=\int_S K'(s, t)\phi(s)d\alpha$ where K' is measurable. In the next theorem a condition is given which is sufficient that such a $K'(s, t)$ exist.

THEOREM 2.4.3. *From the assumptions of Theorem* 2.4.2 *it follows that the associated operation*

$$U'(\phi) = \frac{d}{dt}\int_S H(s, t)\phi(s)d\alpha$$

from $L(S)$ *to* $L(T)$ *can be represented as*

$$U'(\phi) = \int_S \psi_s\phi(s)d\alpha \tag{2.4.31}$$

where $\psi. \in \mathfrak{A}^\infty(S)[L(T)]$, *and as*

$$U'(\phi) = \int_S K'(s, t)\phi(s)d\alpha \tag{2.4.32}$$

where (i) $K'(s, t)$ *is measurable,* (ii) $K'(s, .) = \psi_s$ *is in* $L(T)$ *for every* s, *and* (iii) *for almost every* s, $K'(s, t) = dH(s, t)/dt$ *a.e. in* T. *The norm of* U' *is* $|U'| = \text{ess. sup.}_s \int_T |K'(s, t)| d\beta$.

Let $W(\nu) = \psi$ assign to each $\nu(.) \in C^*(T)$ its derivative function $\psi(t) = d\nu(t)/dt$. Then $U'(\phi) = W(U(\phi)) = W(\int_S \nu_s\phi(s)d\alpha)$ where $\nu. \in \mathfrak{A}^\infty(S)[C^*(T)]$ and $\nu_s = H(s, .)$ a.e. in S. Hence [2] $U'(\phi) = \int_S W(\nu_s)\phi(s)d\alpha$, and $\psi_s \equiv W(\nu_s)$ is in $\mathfrak{A}^\infty(S)[L(T)]$. Thus $U'(\phi) = \int_S \psi_s\phi(s)d\alpha$ and $|U'| = \text{ess. sup.} \|\psi_s\|$. Theorem 1.3.5 now provides a measurable kernel $K'(s, t)$ such that $K'(s, .)$ is in $L(T)$ for each s, $U'(\phi) = \int_S K'(s, t)\phi(s)d\alpha$ for every $\phi \in L(S)$, and $|U'| = \text{ess. sup.}_s \int_T |K'(s, t)| d\beta$. Finally, for almost every s we have $W(H(s, .)) = W(\nu_s) = \psi_s = K'(s, .)$ and so $dH(s, t)/dt = K'(s, t)$ a.e. in T.

The preceding results may be collected in the following statement.

THEOREM 2.4.4. *For a given* $U \in \mathfrak{U}$ *these three conditions are equivalent*:

(2.4.41) *U has a representation* (2.3.1*) *with* $K(s, t)$ *measurable and* $K(s, .)$ *almost separably-valued in* $BV(T)$,

(2.4.42) *U has a representation* (2.3.1) *with* $K(s, .)$ *almost separably-valued in* $C^*(T)$,

(2.4.43) *there is a representation* $U(\phi) = \int_S \nu_s\phi(s)d\alpha$ *with* $\nu. \in \mathfrak{A}^\infty(S)[C^*(T)]$.

Each of these implies that

(2.4.44) $U'(\phi) = dU(\phi)/dt$ *can be written as* $U'(\phi) = \int_S \psi_s\phi(s)d\alpha$ *with* $\psi. \in \mathfrak{A}^\infty(S)[L(T)]$, *and*

(2.4.45) U' *has a representation* $U'(\phi) = \int_S K'(s, t)\phi(s)d\alpha$ *where* $K'(s, t)$ *is measurable,* $K'(s, .)$ *is defined to* $L(T)$, *and for almost every* s, $K'(s, t) = dK(s, t)/dt$ *a.e. in* T, $K(s, t)$ *being any kernel satisfying* (2.4.41) *or* (2.4.42).

If one of the conditions (2.4.41)–(2.4.43) *is satisfied, then* $|U'| = \text{ess. sup.}_s \int_T |K'(s, t)| d\beta$.

Perhaps the only remark needed here is this. Suppose (2.4.41) holds and let $K'(s, t)$ be the kernel given in Theorem 2.4.3. Then for almost every s we have $K'(s, t) = dK(s, t)/dt$ a.e. in T, where $K(s, t)$ is the kernel assumed to satisfy (2.4.41). But if $K_1(s, t)$ is any other kernel satisfying (2.4.41) or (2.4.42), then $K_1(s, t) = K(s, t)$ is an identity in t for almost all s, by Theorem 2.3.8. Thus $K'(s, t)$ fulfills every part of (2.4.45).

B. **Operations U' to $L(T)$ of the form $U'(\phi) = \int_S K'(s, t)\phi(s)d\alpha$.** Thanks to Theorem 2.4.4 several conditions can be given each of which is necessary and sufficient that an operator U' from $L(S)$ to $L(T)$ be representable in the form (2.4.02). In precise terms we have

THEOREM 2.4.5. *Let U' be an operation from $L(S)$ to $L(T)$. The following four conditions are equivalent*:

(2.4.51) *U' can be written as*

$$U'(\phi) = \int_S K'(s, t)\phi(s)d\alpha$$

where K' is measurable and ess. $\sup._s \int_T |K'(s, t)| d\beta < \infty$;

(2.4.52) *U' can be written as*

$$U'(\phi) = \int_S \psi_s \phi(s)d\alpha$$

where $\psi_. \in \mathfrak{A}^\infty(S)[L(T)]$;

(2.4.53) *among the representations*

$$U'(\phi) = \frac{d}{dt}\int_S H(s, t)\phi(s)d\alpha$$

there is one with H measurable and $H(s, .)$ essentially separably-valued in $BV(T)$,

(2.4.54) *among those $U \in \mathfrak{U}$ for which $U'(\phi) = dU(\phi)/dt$ there is one having representation $U(\phi) = \int_S \nu_s \phi(s)d\alpha$ where $\nu_. \in \mathfrak{A}^\infty(S)[C^*(T)]$.*

Under condition (2.4.51) *the norm of U' is* $|U'| =$ ess. $\sup._s \int_T |K'(s, t)| d\beta$.

From Theorem 2.4.4 it is seen that (2.4.53) and (2.4.54) are equivalent and that either implies both (2.4.51) and (2.4.52). Conditions (2.4.51) and (2.4.52) are equivalent in view of Theorems 1.3.5 and 1.3.6.† To complete the proof it is sufficient to show that (2.4.52) implies (2.4.54). Let I be the operator from $L(T)$ to $AC(T)$ assigning to each ψ its indefinite integral. Setting

† Thus when T is abstract (2.4.51) and (2.4.52) are still equivalent, and the norm of U' is ess $\sup_s \int_T |K'(s, t)| d\beta$ in case (2.4.51) is satisfied. The second statement was established by Vulich [42] for real intervals S and T.

$V(\phi)=I(U'(\phi))$ it is evident that $V \varepsilon \mathfrak{U}$ and that $U'(\phi)=dV(\phi)/dt$. Moreover $U(\phi)=I(\int_S\psi_s\phi(s)d\alpha)=\int_S\nu_s\phi(s)d\alpha$, where $\nu_s\equiv I(\psi_s)$ and $\nu. \ \varepsilon\ \mathfrak{A}^\infty(S)[AC(T)]$ since $\psi. \ \varepsilon\ \mathfrak{A}^\infty(S)[L(T)]$. Thus (2.4.54) is satisfied.

Two comments on condition (2.4.51) may be made. First, the requirement that ess. $\sup._s \int_T|K'(s,\ t)|d\beta$ be finite may be replaced by the equivalent one that $K'(s,\ .)$ be essentially defined to $L(T)$ and that ess. $\sup._s\ |\int_T K'(s,\ t)\psi'(t)d\beta|$ be finite for each $\psi'(.)\ \varepsilon\ L^\infty(T)$. Secondly, if S is Euclidean, K' is measurable, and $U'(\phi)=\int_S K'(s,\ t)\phi(s)d\alpha$ is an operation from $L(S)$ to $L(T)$, then necessarily ess. sup. $\int_T|K'(s,\ t)|d\beta<\infty$, as Kantorovitch and Vulich have shown [24, p. 146].

C. **Operations to $AC(T)$ given by almost separably-valued kernels.** Among the operations to $C^*(T)$ that are representable by almost separably-valued kernels there is the proper subclass consisting of those having their range in $AC(T)$. This subclass is characterized in the following theorem.

THEOREM 2.4.6. *For a given $U\ \varepsilon\ \mathfrak{U}$ these five conditions are equivalent*:

(2.4.61) $U(\phi)=\int_S K(s,\ t)\phi(s)d\alpha$ *where K is measurable, $K(s,\ .)$ is almost separably-valued in $BV(T)$, and $\int_E K(s,t)d\alpha\ \varepsilon\ AC(T)$ for every $E\ \varepsilon\ \mathcal{E}_B$,*

(2.4.62) $U(\phi)=\int_S K(s,\ t)\phi(s)d\alpha$ *where $K(s,\ .)$ is essentially defined to $AC(T)$,*

(2.4.63) $U(\phi)=\int_S\nu_s\phi(s)d\alpha$ *with* $\nu.\ \varepsilon\ \mathfrak{A}^\infty(S)[AC(T)]$,

(2.4.64) $U(L(S))\subset AC(T)$ *and* $U'(\phi)=\int_S\psi_s\phi(s)d\alpha$ *with* $\psi.\ \varepsilon\ \mathfrak{A}^\infty(S)[L(T)]$.

(2.4.65) $U(L(S))\subset AC(T)$ *and* $U'(\phi)\doteq\int_S K'(s,\ t)\phi(s)d\alpha$ *where K' is measurable and* ess. $\sup._s \int_T|K'(s,\ t)|d\beta<\infty$.

A kernel K satisfies (2.4.61) *if and only if it satisfies* (2.4.62); *if it satisfies either and if K' fulfills* (2.4.65), *then*

$$|U|=|U'|=\operatorname*{ess.\,sup.}_s \int_T |K'(s,t)|\,d\beta=\operatorname*{ess.\,sup.}_s\,[\mathrm{var}_t\,K(s,t)]$$
$$=\operatorname*{ess.\,sup.}_s \int_T\left|\frac{d}{dt}K(s,t)\right|d\beta$$

and, for almost every s, $K'(s,\ t)=dK(s,\ t)/dt$ a.e. in T.

Suppose (2.4.61) holds for a given kernel K. Theorem 2.4.2 then asserts that $U(\phi)=\int_S\nu_s\phi(s)d\alpha$ where $\nu.\ \varepsilon\ \mathfrak{A}^\infty(S)[C^*(T)]$ and $\nu_s=K(s,\ .)$ a.e. in S. From Theorem 2.4.1 and the assumption that $U(L(S))\subset AC(T)$ it follows that $\nu.\ \varepsilon\ \mathfrak{A}^\infty(S)[AC(T)]$ and hence $K(s,\ .)$ is essentially defined to $AC(T)$. Thus if K fulfills (2.4.61) it also fulfills (2.4.62). When K satisfies (2.4.62) it follows from Theorems 2.4.2. and 2.3.8 that (2.4.63) holds and that K satisfies (2.4.61). Condition (2.4.64) follows easily from (2.3.63); it is sufficient to recall Theorem 1.2.9 and the proof of Theorem 2.4.3. The equivalence of (2.4.64) and (2.4.65) is included in Theorem 2.4.5, and it is obvious that (2.4.65) implies (2.4.63). Finally, (2.4.61) results from (2.4.63) by Theorem 2.4.4. This completes the proof of the equivalence of the conditions (2.4.61)–(2.4.65) and the fact that K fulfills (2.4.61) if and only if it fulfills (2.4.62). The rest of the theorem can be inferred from Theorems 2.4.4 and 2.4.5.

THEOREM 2.4.7. *Let U be in $\mathfrak{U}$ and $U(\phi)=\int_S H(s, t)\phi(s)d\alpha$ where $H(s, .)$ is essentially defined to $C^*(T)$. Suppose any one of the conditions* (2.4.61)–(2.4.65) *is satisfied. If $K(s, t)$ is any kernel fulfilling* (2.4.61) *or* (2.4.62), *then $H(s, .)=K(s, .)$ in $C^*(T)$ for almost all s. Hence $H(s, .)$ is essentially defined to $AC(T)$.*

This comes immediately from Theorems 2.3.8, 2.4.2, and 2.4.6.

By considering two simple examples a few comments can be made on the preceding theorems. Set $S=T=[0, 1]$ and let U assign to each $\phi \in L(S)$ its indefinite integral. It is easily verified that $U(\phi)=\int_S H(s, t)\phi(s)d\alpha$ where $H(s, t)=0$ if $s>t$ and $H(s, t)=1$ if $s\leqq t$. From this and Theorem 2.4.7 there follows the well known fact that the identity operation I from $L(S)$ to $L(S)$ cannot be represented as $I(\phi)=\phi=\int_S K'(s, t)\phi(s)d\alpha$ where $K'(s, t)$ is measurable. For here U' coincides with I, and if I were so representable (i) and (ii) of (2.4.65) would be fulfilled; it would then result that $H(s, .)$ is essentially defined to $AC(T)$, which is clearly false. This example also shows that those operators from $L(S)$ to $AC(T)$ defined by almost separably-valued kernels form a proper subclass of all the operators on $L(S)$ to $AC(T)$. Consequently the set of all operations from $L(S)$ to $C^*(T)$, $S=T=[0, 1]$, is larger than the subclass consisting of those having representations by means of almost separably-valued kernels.

The operators characterized in Theorem 2.4.6 form an even smaller class. For an example it is enough to consider $U(\phi)=\int_S \eta(t)\xi(s)\phi(s)d\alpha$ where $S=T=[0, 1]$, $\eta(.) \in C^*(T)-AC(T)$, and $\xi(.)$ is bounded, measurable, and has $\infty >$ess. sup. $|\xi(s)|>0$. Here $AC(T) \not\supset U(L(S))$, yet $H(s, .)\equiv\xi(s)\eta(.)$ is separably-valued and is in $\mathfrak{A}^\infty(S)[C^*(T)]$.

These two examples also prove that no one of the conditions (2.4.61)–(2.4.65) remains equivalent to the others if in that particular condition either $AC(T)$ is replaced by $C^*(T)$ or the assumption that $H(s, .)$ is almost separably-valued is dropped.

PART 5. THE ITERATIVE INTEGRALS ASSOCIATED WITH CERTAIN KERNELS

Before ending Chapter II we should like to make a few remarks concerning the existence and equality of the iterative integrals associated with some of the kernels discussed in Parts 3 and 4. The following result, already established in Theorem 2.3.7, is an illustration.

THEOREM 2.5.1. *Suppose U is in $\mathfrak{U}$ and $U(\phi)=\int_S H(s,t)\phi(s)d\alpha$ where* (i) *$H(s,t)$ is measurable and* (ii) *$H(s,t)$ is everywhere continuous on the right in t. Then the two mixed integrals*

$$\mathrm{S}\int_T \gamma(t)d\left\{\int_S H(s,t)\phi(s)d\alpha\right\}, \qquad \int_S\left\{\mathrm{S}\int_T \gamma(t)dH(s,t)\right\}\phi(s)d\alpha$$

exist and are equal when $\gamma(.)\ \varepsilon\ C(T)$ and $\phi(.)\ \varepsilon\ L(S)$.

Another result of the same sort is

THEOREM 2.5.2. *Theorem* 2.5.1 *remains true if in place of* (ii) *it is assumed that* $\operatorname{var}_t H(s,t)<\infty$ *for almost every s.*

Let $K(s,t)$ be the kernel given in the conclusions of Theorem 2.3.6; the above two integrals then necessarily exist and coincide when H is replaced by K. In addition, $\int_S H(s,t)\phi(s)d\alpha=\int_S K(s,t)\phi(s)d\alpha$, so that

$$\mathrm{S}\int_T \gamma(t)d\left\{\int_S H(s,t)\phi(s)d\alpha\right\} = \int_S\left\{\mathrm{S}\int_T \gamma(t)dK(s,t)\right\}\phi(s)d\alpha.$$

On the other hand, for almost every s the two functions $H(s,t)$ and $K(s,t)$ are of bounded variation with respect to t and coincide a.e. in T. This implies that $\mathrm{S}\int_T\gamma(t)dH(s,t)=\mathrm{S}\int_T\gamma(t)dK(s,t)$ for every $\gamma(.)\ \varepsilon\ C(T)$ and almost every s, and hence

$$\int_S\left\{\mathrm{S}\int_T \gamma(t)dH(s,t)\right\}\phi(s)d\alpha = \int_S\left\{\mathrm{S}\int_T \gamma(t)dK(s,t)\right\}\phi(s)d\alpha.$$

For almost separably-valued kernels a stronger conclusion can be made.

THEOREM 2.5.3. *If, in Theorem* 2.5.1, (ii) *is replaced by the assumption that $H(s,\ .)$ is almost separably-valued in $BV(T)$, then the two mixed integrals (Lebesgue and Radon-Stieltjes)*

$$\int_S \phi(s)\left\{\int_Q H(s,t)dF(P)\right\}d\alpha, \qquad \int_Q\left\{\int_S H(s,t)\phi(s)d\alpha\right\}dF(P)$$

exist and are equal for each $\phi\ \varepsilon\ L(S)$ and each $F(P)$ which is bounded and additive over all subsets P of the space Q consisting of all partitions of T.

From Theorem 2.4.2 the function $\nu_s\equiv H(s,\ .)$ is in $\mathfrak{A}^\infty(S)[C^*(T)]$ and $U(\phi)=\int_S\nu_s\phi(s)d\alpha$. For each linear functional $f(\nu)$ over $C^*(T)$ we then have $f(U(\phi))=\int_S f(\nu_s)\phi(s)d\alpha$. Since each $F(P)$ defines such a functional by means of the Radon-Stieltjes integral [20] and since $\nu_s=H(s,\ .)$ in $C^*(T)$ for almost every s, it follows that $\int_Q\{\int_S H(s, t)\phi(s)d\alpha\}dF(P) = f(U(\phi)) = \int_S\{\int_Q H(s, t)dF(P)\}\phi(s)d\alpha$.

THEOREM 2.5.4. *From the assumptions of Theorem* 2.5.3 *it also follows that*

$$\int_T \psi(t)\left[\frac{d}{dt}\int_S H(s, t)\phi(s)d\alpha\right]dt = \int_S \phi(s)\left[\int_T \psi(t)\left(\frac{d}{dt} H(s, t)\right)dt\right]d\alpha$$

whenever $\phi \varepsilon L(S)$ *and* $\psi \varepsilon L^\infty(T)$. *Hence*

$$\int_c^\tau \left[\frac{d}{dt}\int_E H(s, t)d\alpha\right]dt = \int_E\left[\int_c^\tau\left(\frac{d}{dt} H(s, t)\right)dt\right]d\alpha$$

for each $\tau \varepsilon T$ *and each* $E \varepsilon \mathcal{E}_B$.

Here the operation $U'(\phi) = (d/dt)\int_S H(s, t)\phi(s)d\alpha$ can be written as $U'(\phi) = \int_S \psi_s\phi(s)d\alpha$ where $\psi. \varepsilon \mathfrak{A}^\infty(S)[L(T)]$ and ψ_s is for almost every s the element $dH(s, t)/dt$ in $L(T)$. Thus $f(U'(\phi)) = \int_S f(\psi_s)\phi(s)d\alpha$ for every f in the adjoint of $L(T)$, so that the above two integrals exist and are equal for each $\phi \varepsilon L(S)$ and $\psi \varepsilon L^\infty(T)$.

CHAPTER III. RESTRICTED OPERATIONS TO ARBITRARY SPACES

PART 1. WEAKLY COMPLETELY CONTINUOUS AND COMPLETELY CONTINUOUS OPERATIONS

In this chapter the restrictions and emphasis are placed upon the operation rather than upon the range space. Only those operations U will be considered which map $L(S)$ into an arbitrary B-space X and fulfill one of the requirements (i) U is defined by an element of $\mathfrak{A}^\infty(S)[X]$, (ii) U is weakly c.c., or (iii) it is c.c. Our aim is to obtain properties and vector integral representations of the three classes of operators defined respectively by these conditions.

Part 1 contains the proof that every U satisfying (i) must map weakly compact sets into compact sets and also provides conditions necessary and sufficient that an operation which satisfies (i) also satisfy (ii) or (iii). When S is Euclidean we are able to be more precise; in Theorem 3.1.9 characteristic representations are given for the weakly c.c. and the c.c. operations from $L(S)$ to* X. In addition it is shown, without restricting S or T, that a mapping U is a c.c. operator from $L(S)$ to $L^q(T)$, $1 \leqq q \leqq \infty$, if and only if $U(\phi) = \int_S x(s)\phi(s)d\alpha$ where $x(.)$ is an essentially compact-valued element of $\mathfrak{A}^\infty(S)[L^q(T)]$.

* The representation to be given for c.c. operations was originally due to Gelfand [18], who established it for the case in which S is a bounded real interval.

A. **Sufficient conditions for weak complete continuity and complete continuity.** We begin by considering those operations generated by certain abstract integrals.

THEOREM 3.1.1. *Let* $x(.)$ *be in* $\mathfrak{L}_0^\infty(S)[X, X^*]$. *Then* (i) *if* $x(.)$ *is essentially weakly compact-valued, it is essentially bounded and the operation* $U(\phi) = \int_S x(s)\phi(s)d\alpha$ *is weakly c.c. from* $L(S)$ *to* X; *and* (ii) *if* $x(.)$ *is essentially compact-valued, it is in* $\mathfrak{A}^\infty(S)[X]$ *and* U *is c.c.*

Suppose $E_0 \in \mathcal{E}_0$ is such that $x(S-E_0)$ is weakly compact. Every weakly compact set being bounded, $x(s)$ obviously is essentially bounded. Moreover, $R(x(S-E_0))$ must be weakly compact, so that the point set sum $Y = x(S-E_0) + R(x(S-E_0))$ has the same property. From a theorem of Chmoulyan [39] it follows that the closed convex hull $C[Y]$ is necessarily weakly compact. Since $C[Y] \ni U(\phi)$ for every ϕ with $\|\phi\| \leqq 1$ by part (iii) of Theorem 1.2.5, clearly U is weakly c.c. If $x(S-E_0)$ satisfies the stronger condition of being compact, then $x(.)$ is essentially bounded and almost separably-valued, and is therefore in $\mathfrak{A}^\infty(S)[X]$ by Theorem 1.1.7 since $f(x(s))$ is in $L^\infty(S)$ for every $f \in X^*$. In addition the set Y is compact and so $C[Y]$ is compact, from a result of Mazur [28]. Again applying Theorem 1.2.5 it is seen that U is c.c.

The next theorem is one of the most important in the paper.

THEOREM 3.1.2. *For an otherwise arbitrary* $x(s)$ *suppose there exist an* $E_0 \in \mathcal{E}_0$ *and a c.l.m.* $\Gamma \subset X^*$ *such that* $x(S-E_0)$ *is separable and* Γ *is a determining manifold for* $x(S-E_0)$. *If* $x(.) \in \mathfrak{L}^\infty(S)[X, \Gamma]$, *then* $x(.) \in \mathfrak{A}^\infty(S)[X]$. *Moreover the following are true for* $x(.)$ *since they are true for any element of* $\mathfrak{A}^\infty(S)[X]$:

(I) $U(\phi) = \int_S x(s)\phi(s)d\alpha$ *defines a separable operation* U *from* $L(S)$ *to* X *that takes weakly compact sets into compact sets and has* $|U| = \text{ess. sup. } \|x(s)\|$,

(II) U *is weakly c.c. if and only if* $x(.)$ *is essentially weakly compact-valued,*

(III) U *is c.c. if and only if* $x(.)$ *is essentially compact-valued.*

The first statement is given by Theorem 1.1.7. Now suppose $x(.)$ is any element of $\mathfrak{A}^\infty(S)[X]$. Since conclusions (II) and (III) follow immediately from Theorem 3.1.1 and (IV) of Theorem 1.2.10, only (I) remains to be proved. We thus wish to show that $U(\Phi)$ is compact when Φ is weakly compact and $x(.)$ is an arbitrary element in the class $\mathfrak{A}^\infty(S)[X]$.

Suppose $x(.)$ is almost countably-valued, that is,

(1) in $\mathcal{E}$ there are disjoint elements E_i, $i=0, 1, 2, \cdots$, such that $\sum_0^\infty E_i = S$, $E_0 \in \mathcal{E}_0$, and, for $i \geqq 1$, $x(s)$ has a constant value x_i on E_i.

For each $\phi \varepsilon L(S)$ we have $\sum_1^\infty |\int_{E_i}\phi(s)d\alpha| \leqq \|\phi\|$ and hence $P(\phi) = \{\int_{E_i}\phi(s)d\alpha\}$, $(i=1, 2, \cdots)$, is an operation from $L(S)$ to l. The image $P(\Phi)$ of the weakly compact set Φ must then be weakly compact. Since in l compactness and weak compactness are equivalent properties, the set $P(\Phi)$ is actually compact. Now consider the bounded sequence $\{x_i\}$. The mapping $Q(\psi) = \sum_1^\infty x_i\psi_i$ is defined from l to X and is an operation, so that $Q(P(\Phi))$ is compact in X. But for each $\phi \varepsilon L(S)$ the equality $U(\phi) = \sum_1^\infty \int_{E_i} x(s)\phi(s)d\alpha = \sum_1^\infty x_i \int_{E_i}\phi(s)d\alpha$ holds, and hence $U(\phi) = Q(P(\phi))$. Thus U takes weakly compact sets into compact sets when $x(.)$ satisfies (1).

Now let $x(.)$ in $\mathfrak{A}^\infty(S)[X]$ be arbitrary and let K be a bound for the weakly compact set Φ. From the measurability of $x(.)$ it follows [31, Corollary 1.12] that for each $\epsilon > 0$ a measurable function $x'(s)$ exists satisfying both (1) and

(2) ess. sup. $\|x(s) - x'(s)\| < \epsilon/2K$.

From this fact we can conclude that $U(\Phi)$ is totally bounded and hence is compact. For $x'(.) \varepsilon \mathfrak{A}^\infty(S)[X]$ as a consequence of (2), so that by the preceding case (1) implies that $U'(\phi) = \int_S x'(s)\phi(s)d\alpha$ is an operation mapping Φ into a compact set. Thus $U'(\Phi)$ is totally bounded and can be covered by a finite number of spheres having centers x_i', $i=1, 2, \cdots, m$, and radii each less than $\epsilon/2$. Since

$$\|U(\phi) - U'(\phi)\| \leqq \text{ess. sup. } \|x(s) - x'(s)\| \cdot \|\phi\|, \qquad \phi \varepsilon L(S),$$

we have $\|U(\phi) - U'(\phi)\| < \epsilon/2$ for $\phi \varepsilon \Phi$. This implies that for each $\phi \varepsilon \Phi$ the point $U(\phi)$ is within ϵ of one of the centers x_i'. Thus $U(\Phi)$ is totally bounded.

The next four theorems are immediate corollaries to Theorem 3.1.2.

THEOREM 3.1.3. *A given measurable function $x(s)$ is essentially (weakly) compact-valued if and only if $U(\phi) = \int_S x(s)\phi(s)d\alpha$ is defined and (weakly) c.c. from $L(S)$ to X.*

THEOREM 3.1.4. *A necessary and sufficient condition that $\{x_n\} \subset X$ be a (weakly) compact sequence is that $\{x_n\}$ is bounded and the operation $U(\phi) = \sum_n \phi_n x_n$, $\phi = \{\phi_n\}$, from l to X is (weakly) c.c.**

THEOREM 3.1.5. *U is defined and (weakly) c.c. from l to X if and only if if it can be written as $U(\phi) = \sum_n \phi_n x_n$ where $\{x_n\}$ is (weakly) compact in X. The norm of U is*† $\sup \|x_n\|$.

THEOREM 3.1.6. *If $x(.) \varepsilon \mathfrak{A}^\infty(S)[X]$ is essentially weakly compact-valued, then $U(\phi) = \int_S x(s)\phi(s)d\alpha$ is weakly c.c. from $L(S)$ to X and takes weakly com-*

* The two propositions given in Theorem 3.1.4 are equivalent respectively to those of Chmoulyan and Mazur used in proving Theorem 3.1.1.

† The statement concerning c.c. operations that is included in this theorem is due to Gelfand [18].

pact sets into compact sets. Hence, if $X = L(S)$, the iterated operation $U^n(\phi)$ is defined and c.c. from $L(S)$ to $L(S)$ for $n \geqq 2$, which implies that the fixed points of U form a finite-dimensional c.l.m. in $L(S)$.

From (I) of Theorem 3.1.2 and Theorems 2.1.8 and 2.1.9 we also have

THEOREM 3.1.7. *If X is either a separable adjoint space or has a weakly compact unit sphere, then*

(i) *U is an operation defined and c.c. from $L(S)$ to X if and only if $U(\phi) = \int_S x(s)\phi(s)d\alpha$ where $x(.)$ is an essentially compact-valued element of $\mathfrak{A}^\infty(S)[X]$.*

If X has a weakly compact unit sphere, then

(ii) *U is an operation defined, separable, and weakly c.c. from $L(S)$ to X if and only if $U(\phi) = \int_S x(s)\phi(s)d\alpha$ where $x(.)$ is an essentially weakly compact-valued element of $\mathfrak{A}^\infty(S)[X]$, and*

(iii) *every separable operation from $L(S)$ to X takes weakly compact sets into compact sets.*

When X is a separable adjoint space, statements (ii) *and* (iii) *are true with the word "separable" deleted.*

The space $L^q(T)$ having a weakly compact unit sphere in case $1 < q < \infty$, Theorem 3.1.7 yields

THEOREM 3.1.8. *When $X = L^q(T)$, $1 < q < \infty$, conclusions* (i)–(iii) *of Theorem* 3.1.7 *hold.*

B. **Criteria for weak complete continuity and complete continuity when S is Euclidean.** The theorem below solves for Euclidean S the problems of characterizing the weakly c.c. and the c.c. operations from $L(S)$ to an arbitrary X.

THEOREM 3.1.9. *Let S be a Euclidean interval, finite or infinite, and let α be Lebesgue measure. An operation U is defined and (weakly) c.c. from $L(S)$ to X if and only if there exists in $\mathfrak{A}^\infty(S)[X]$ an essentially (weakly) compact-valued $x(.)$ such that $U(\phi) = \int_S x(s)\phi(s)d\alpha$. The norm of U is* ess. sup. $\|x(s)\|$.

By reason of Theorem 3.1.2 and the fact that complete continuity implies weak complete continuity, it suffices to prove that every weakly c.c. operation U from $L(S)$ to X has a representation $U(\phi) = \int_S x(s)\phi(s)d\alpha$ where $x(.) \in \mathfrak{A}^\infty(S)[X]$. Suppose U to be such an operation and S to be bounded. Consider a fixed point s in the interior S^0 of S and let $\{I_n'\}$ be some sequence of nondegenerate cubes such that $S^0 \supset I_n' \ni s$ and $\lim_n \alpha(I_n') = 0$. If $x_n' = U(\phi_n')$ where $\phi_n' = \phi_{I_n'}/\alpha(I_n')$ then since $\|\phi_n'\| \leqq 1$ and U is weakly c.c. it follows that $\{x_n'\}$ contains a subsequence $\{x_n\}$ converging weakly to an element $x(s)$ of X. From this convergence property it is evident for any $f \in X^*$ that

$f(x(s)) = \lim_n f(x_n) = \lim_n f(X_{I_n})/\alpha(I_n)$ where X_R is the image under U of the characteristic function of the figure R. Since $\{I_n\}$ closes down on s, this implies that if $f(X_R)$ is differentiable at s its derivative there has the value $f(x(s))$. Since $\alpha(S-S^0)=0$, the function $x(s)$ is thus defined a.e. in S and has the property that for each $f \varepsilon X^*$ the numerical additive Lipschitzean function $f(X_R)$ has its derivative coinciding with $f(x(s))$ wherever this derivative exists. At this point we make the observation that since $L(S)$ is separable and U is continuous, there is no loss of generality in assuming X to be separable. We now have the following situation: (i) X_R is an additive Lipschitzean function defined to X from the figures lying in the bounded Euclidean interval S, (ii) $x(s)$ is a point function defined a.e. in S and having its values in the separable space X, and (iii) for each $f \varepsilon X^*$ the numerical function $f(X_R)$ is differentiable a.e. in S to the value $f(x(s))$. From Theorem 2.7 of [32] it follows immediately that $x(.)$ is measurable and integrable and that X_R is differentiable a.e. in S to the value $x(s)$. Since X_R is additive and Lipschitzean, we see that $X_R = \int_R x(s)d\alpha$ and $x(.) \varepsilon \mathfrak{A}^\infty(S)[X]$. Thus $U(\phi) = \int_S x(s)\phi(s)d\alpha$ where $x(.) \varepsilon \mathfrak{A}^\infty(S)[X]$.

In case S is unbounded let $\{S_i\}$ be a decomposition of S into bounded intervals. By the preceding case there is for each i an $x_i(s)$ defined a.e. in S_i and such that $U(\phi) = \int_{S_i} x_i(s)\phi(s)d\alpha$ for each $\phi(.) \varepsilon L(S)$ that vanishes a.e. in $S-S_i$; moveover, $x_i(s)$ is measurable in S and $|U| = \text{ess. sup. } \|x_i(s)\|$. Let $x(s) = x_i(s)$ when $s \varepsilon S_i$ and $x_i(s)$ is defined at s. Clearly $x(.) \varepsilon \mathfrak{A}^\infty(S)[X]$, and since any simple function $\phi(s)$ can be written as $\phi(s) = \sum_1^n \phi_i(s)$, where each $\phi_i(s)$ vanishes outside of S_i, we also have $U(\phi) = \sum_1^n U(\phi_i) = \sum_1^n \int_{S_i} x_i(s)\phi_i(s)d\alpha = \sum_1^n \int_S x(s)\phi_i(s)d\alpha = \int_S x(s)\phi(s)d\alpha$ whenever $\phi(s)$ is simple. This of course implies that $U(\phi) = \int_S x(s)\phi(s)d\alpha$ for every $\phi \varepsilon L(S)$.

COROLLARY. *When S is Euclidean and α is Lebesgue measure, any weakly c.c. operation U from $L(S)$ to X has the property of taking weakly compact sets into compact sets. Hence if $X = L(S)$ the nth iterate U^n is c.c. for $n \geqq 2$ and thus U has for its fixed points a finite-dimensional c.l.m.*

C. **Abstract representations of completely continuous operations to $L^q(T)$, $1 \leqq q \leqq \infty$.** We turn now to the proof that the characterization of c.c. operators given in the last theorem is also true for unrestricted S provided that X is an $L^q(T)$ space.

THEOREM 3.1.10. *An operation U is defined and c.c. from $L(S)$ to $L^q(T)$, $1 \leqq q \leqq \infty$, if and only if $U(\phi) = \int_S x(s)\phi(s)d\alpha$ where $x(.)$ is an essentially compact-valued element of $\mathfrak{A}^\infty(S)[L^q(T)]$. The norm of U is $|U| = \text{ess. sup. } \|x(s)\|$.*

The case in which $1 < q < \infty$ has been covered in Theorem 3.1.8. In virtue of Theorem 3.1.2 all that is required then is the proof that each c.c. operation from $L(S)$ to $L^q(T)$, $q=1$ or ∞, can be represented by some element of $\mathfrak{A}^\infty(S)[L^q(T)]$. The case $q=1$ will be considered first.

Let π stand for a finite number of disjoint measurable sets Δ in T such that $\infty > \beta(\Delta) > 0$, $\Delta \varepsilon \pi$. If $\pi = (\Delta)$ and $\pi' = (\Delta')$ are two such partitionings, define $\pi\pi'$ to be the partitioning consisting of those $\Delta\Delta'$ with $\beta(\Delta\Delta') > 0$. We shall use the relation $\pi \subset \pi'$ to mean that for every $\Delta \varepsilon \pi$ there is a $\Delta' \varepsilon \pi'$ with $\beta(\Delta - \Delta') = 0$. It is clear then that $\pi\pi' \subset \pi$. If x and $x(\pi)$ are points in a B-space, the statement

$$\lim_{\pi} x(\pi) = x \tag{1}$$

means that for every $\epsilon > 0$ there is a π_ϵ such that $\|x(\pi) - x\| < \epsilon$ if $\pi \subset \pi_\epsilon$. If (1) holds, there is a sequence π_n with $x(\pi_n) \to x$. Similarly we say that $x(\pi)$ is convergent if

$$\|x(\pi) - x(\pi')\| < \epsilon \qquad \text{for } \pi, \pi' \subset \pi_\epsilon.$$

If (1) holds, $x(\pi)$ is convergent and if $x(\pi)$ is convergent there is a unique x for which (1) is true.

Let Y_π be the c.l.m. in $L(T)$ consisting of those elements which are constant on each of the sets $\Delta \varepsilon \pi$. Thus

$$Y_{\pi'} \subset Y_\pi \qquad \text{if } \pi \subset \pi'. \tag{2}$$

Let ψ_Δ be the characteristic function of Δ and U_π the projection of $L(T)$ into Y_π which is defined by

$$U_\pi(\psi) = \sum_{\Delta \varepsilon \pi} \psi_\Delta \frac{\int_\Delta \psi(t) d\beta}{\beta(\Delta)}. \tag{3}$$

With $U_\pi(\psi)$ thus defined we have

$$|U_\pi| = 1, \qquad U_\pi(\psi) = \psi \qquad \text{for } \psi \varepsilon Y_\pi. \tag{4}$$

From (2) and (4) we see that for a simple function ψ', say $\psi' \varepsilon Y_{\pi'}$, we have $U_\pi(\psi') = \psi'$ for $\pi \subset \pi'$ and thus for simple functions

$$\lim_{\pi} U_\pi(\psi) = \psi. \tag{5}$$

Since the simple functions are dense in $L(T)$ and $|U_\pi| = 1$, it follows that (5) holds for every $\psi \varepsilon L(T)$. Moreover (5) holds uniformly over any compact set. To see this we take points ψ_i, $i = 1, 2, \cdots, n_\epsilon$, in a compact set Ψ such that every point in Ψ is within ϵ distance of some ψ_i. There is then a π_ϵ such that

$$\|U_\pi(\psi_i) - \psi_i\| < \epsilon, \qquad i = 1, 2, \cdots, n_\epsilon, \pi \supset \pi_\epsilon,$$

and thus

$$\|U_\pi(\psi) - \psi\| < 3\epsilon \qquad \text{for } \psi \in \Psi, \pi \supset \pi_\epsilon.$$

Hence if U is c.c. from $L(S)$ into $L(T)$ we have

$$\lim_\pi |U_\pi U - U| = 0.$$

There is then a sequence π_n such that

$$(6) \qquad \lim_n |U_{\pi_n} U - U| = 0, \qquad \lim_{m,n} |U_{\pi_m} U - U_{\pi_n} U| = 0.$$

For a c.c. operation $\psi = U(\phi)$ on $L(S)$ to $L(T)$ we have (see (3))

$$U_\pi(U(\phi)) = \sum_{\Delta \in \pi} f_\Delta(\phi)\psi_\Delta,$$

where $f_\Delta(\phi) = \int_\Delta \psi(t) d\beta / \beta(\Delta)$ is a linear functional on $L(S)$. Thus there are elements $\mu_\Delta(.) \in L^\infty(S)$ such that

$$(7) \qquad U_\pi(U(\phi)) = \sum_{\Delta \in \pi} \psi_\Delta \int_S \mu_\Delta(s)\phi(s)d\alpha = \int_S x_\pi(s)\phi(s)d\alpha$$

where $x_\pi(.)$ is the abstract function $x_\pi(s) = \sum_{\Delta \in \pi} \psi_\Delta \mu_\Delta(s)$. Writing $x_n(s)$ for $x_{\pi_n}(s)$, it is clear that $x_n(.) \in \mathfrak{A}^\infty(S)[L(T)]$. From (6), (7), and Theorem 1.1.7

$$\lim_{m,n} \text{ess. sup.} \|x_m(s) - x_n(s)\| = 0,$$

and hence $x(s) = \lim_m x_m(s)$ is defined a.e. and $x(.) \in \mathfrak{A}^\infty(S)[L(T)]$, with

$$(8) \qquad \lim_m \text{ess. sup.} \|x_m(s) - x(s)\| = 0.$$

Thus $U'(\phi) = \int_S x(s)\phi(s)d\alpha$ is an operation from $L(S)$ to $L(T)$ and (8) shows that $\lim_m |U_{\pi_m} U - U'| = 0$. Combining this with (6) we have $U = U'$ and hence $U(\phi) = \int_S x(s)\phi(s)d\alpha$ where $x(.) \in \mathfrak{A}^\infty(S)[L(T)]$.

The proof for $q = \infty$ is somewhat similar. Let U be c.c. from $L(S)$ to $L^\infty(T)$ and suppose to begin with that $\beta(T) < \infty$. The space $L^\infty(T)$ can then be considered as a linear subset Y of $L(T)$, and the "identity" transformation $I(\mu) = \mu$ is a continuous 1-1 operation taking $L^\infty(T)$ into Y. Since convergence in the mean implies convergence in measure, the operation I is a homeomorphism on each closed compact set in $L^\infty(T)$. Set $V(\phi) = I(U(\phi))$ and let Φ be the unit sphere in $L(S)$. The closure M of $U(\Phi)$ is compact and hence $I(M)$ is closed and compact since I is necessarily a homeomorphism on M. In addition the compactness of $U(\Phi)$ implies that $I(M)$ is the closure of

$V(\Phi)$. The operation V to $L(T)$ clearly being c.c., we have by the preceding case that $V(\phi)=\int_S y(s)\phi(s)d\alpha$ where $y(.)\ \varepsilon\ \mathfrak{A}^\infty(S)[L(T)]$. From Theorem 1.2.10 it follows that $y(.)$ is essentially defined to the closure of $V(\Phi)$, that is, to $I(M)$. But M is closed and compact so that $I(M)$ maps into the closed compact set M under the inverse of I. Hence $x(s)=I^{-1}(y(s))$ is essentially compact-valued. It is therefore essentially bounded and almost separably-valued. To show that $x(.)\ \varepsilon\ \mathfrak{A}^\infty(S)[L^\infty(T)]$ it is enough to prove that $g_\psi(x(s))$ is measurable for each linear functional g_ψ over $L^\infty(T)$ defined by an element ψ of $L(T)$. Let $F\ \varepsilon\ \mathcal{J}_B$ be arbitrary. If g_F is the functional defined by the characteristic function of F, we have $g_F(x(s))=\int_F K(s,t)d\beta$ where $K(s,\ .)=x(s)$ in $L^\infty(T)$ for each s. But then $K(s,\ .)=y(s)$ in $L(T)$ for each s, and therefore $g_F(x(s))=f_F(y(s))$ where f_F is the linear functional over $L(T)$ defined by the characteristic function of F. Since the vector function $y(.)$ is measurable, we conclude that $g_F(x(s))$ is measurable for each $F\ \varepsilon\ \mathcal{J}_B$. The extension to g_ψ where ψ is a simple function is obvious, and, since simple functions are dense in $L(T)$, $g_\psi(x(s))$ must be measurable for every $\psi\ \varepsilon\ L(T)$. Thus $x(.)\ \varepsilon\ \mathfrak{A}^\infty(S)[L(T)]$. Let $U'(\phi)=\int_S x(s)\phi(s)d\alpha$. Then $g_F(U'(\phi))=\int_S g_F(x(s))\phi(s)d\alpha$ for each $F\ \varepsilon\ \mathcal{J}_B$. We also have $f_F(V(\phi))=\int_S f_F(y(s))\phi(s)d\alpha$. Since $f_F(y(s))=g_F(x(s))$ for every s and F, clearly $g_F(U'(\phi))=f_F(V(\phi))$. But obviously $f_F(V(\phi))=g_F(U(\phi))$, I being the "identity." Hence $g_F(U'(\phi))=g_F(U(\phi))$ for every $\phi\ \varepsilon\ L(S)$ and $F\ \varepsilon\ \mathcal{J}_B$. Since the g_F's are a total set of functionals over $L^\infty(T)$, the last equality yields $U'(\phi)=U(\phi)$, that is, $U(\phi)=\int_S x(s)\phi(s)d\alpha$ where $x(.)\ \varepsilon\ \mathfrak{A}^\infty(S)[L^\infty(T)]$. This concludes the case $\beta(T)<\infty$.

In the general case let $\{T_i\}$ be a decomposition of T and let X_n be the c.l.m. in $L^\infty(T)$ composed of all the elements which vanish a.e. in $T-\sum_1^n T_i$. If Π_n assigns to each μ in $L^\infty(T)$ the element of X_n which coincides with $\mu(t)$ a.e. in $\sum_1^n T_i$, clearly Π_n is an operation in $L^\infty(T)$ with $|\Pi_n|=1$. Suppose U is c.c. from $L(S)$ to $L^\infty(T)$. Set $V_n(\phi)=\Pi_n(U(\phi))$. Then V_n is c.c. to X_n, and since X_n is equivalent to $L^\infty(\sum_1^n T_i)$ where $\beta(\sum_1^n T_i)<\infty$ it follows from the preceding case that $V_n(\phi)=\int_S x_n(s)\phi(s)d\alpha$ where $x_n(.)\ \varepsilon\ \mathfrak{A}^\infty(S)[X_n]$. According to Theorem 1.2.10 $x_n(.)$ is essentially defined to the closure of $V_n(\Phi)$. Since $V_n(\Phi)=\Pi_n(U(\Phi))$ where $U(\Phi)$ is compact, we can infer that

$$x_n(.) \text{ is essentially defined to } \Pi_n(M), \qquad n=1,2,\cdots, \tag{9}$$

where M is the bounded compact closure of $U(\Phi)$. Going back to the definition of Π_j it is clear that for any $\mu\ \varepsilon\ L^\infty(T)$

$$\Pi_h\Pi_i(\mu) = \Pi_j(\mu) \qquad \text{for } h,\ i \geqq j. \tag{10}$$

Thus when $j\leqq h,\ i$ we have $V_j(\phi)=\Pi_h\Pi_i(U(\phi))=\Pi_h(V_i(\phi))=\int_S\Pi_h(x_i(s))\phi(s)d\alpha$ where $\Pi_h(x_i(.))\ \varepsilon\ \mathfrak{A}^\infty(S)[L^\infty(T)]$. But V_j is also defined by the element $x_j(.)$

of $\mathfrak{A}^\infty(S)[L^\infty(T)]$. Hence

$$x_j(s) = \Pi_h(x_i(s)) \text{ a.e. in } S, \qquad \text{if } j \leqq h, i. \tag{11}$$

From (9) and (11) we can now draw this conclusion: there is a null set E_0 such that if $s \varepsilon S - E_0$ and h, i, j, n are any integers then

$$x_n(s) \varepsilon \Pi_n(M), \tag{12}$$

$$x_j(s) = \Pi_h(x_i(s)), \qquad \text{if } j \leqq h, i. \tag{13}$$

From (12) we also have

$$\|x_n(s)\| \leqq |U|, \qquad s \varepsilon S - E_0. \tag{14}$$

Fix s in $S - E_0$ and let $x_n = x_n(s)$. Due to (12) there is for each n an element μ_n in the closed compact set M such that $x_n = \Pi_n(\mu_n)$. If $i \geqq j$ we then have

$$\Pi_j(\mu_i) = \Pi_i\Pi_j(\mu_i) = \Pi_j\Pi_i(\mu_i) = \Pi_j(x_i)$$

by (10), so that

$$\Pi_j(\mu_i) = x_j \qquad \text{for } i \geqq j, \tag{15}$$

from (13). Now let x_0 be that measurable function of t which coincides over each T_n with x_n. In view of (14), x_0 is in $L^\infty(T)$, and from (13) and the definition of x_0 it is easily verified that

$$\Pi_j(x_0) = x_j, \qquad j = 1, 2, \cdots. \tag{16}$$

In the compact sequence $\{\mu_n\}$ choose a subsequence $\{\mu_{n_i}\}$ converging to $\mu_0 \varepsilon M$. Fixing j we have $\Pi_j(\mu_0) = \lim_i \Pi_j(\mu_{n_i})$ or, from (15), $\Pi_j(\mu_0) = \lim_i x_j = x_j$. Combining this with (16) yields

$$\Pi_j(x_0) = \Pi_j(\mu_0), \qquad j = 1, 2, \cdots,$$

which implies that $x_0 = \mu_0$. Thus $x_0 \varepsilon M$.

For each $s \varepsilon S - E_0$ define $x_0(s)$ to be this point x_0 and for $s \varepsilon E_0$ let $x_0(s)$ vanish. Since $x_0(.)$ is essentially defined to the compact set M, it is essentially bounded and almost separably-valued. To prove that $x_0(.) \varepsilon \mathfrak{A}^\infty(S)[L^\infty(T)]$ it is therefore sufficient to obtain measurability for $f_\psi(x_0(s))$ whenever f_ψ is a linear functional over $L^\infty(T)$ defined by an element ψ of $L(T)$. For a functional of this sort it is obvious that

$$f_\psi(\mu) = \lim_n \int_{\Sigma_1^n T_i} \psi(t)\mu(t)d\beta = \lim_n f_\psi(\Pi_n(\mu)) \tag{17}$$

for every $\mu \varepsilon L^\infty(T)$. This together with (16) yields the equality $f_\psi(x_0(s)) = \lim_n f_\psi(\Pi_n(x_0(s))) = \lim_n f_\psi(x_n(s))$ for almost all s. Since each $x_n(.)$ is measurable, $f_\psi(x_0(s))$ must therefore be measurable. Thus $x_0(.) \varepsilon \mathfrak{A}^\infty(S)[L^\infty(T)]$.

The final assertion is that $U(\phi)=\int_S x_0(s)\phi(s)d\alpha$. For (17) leads to the equality

$$f_\psi(U(\phi)) = \lim_n f_\psi(V_n(\phi)) = \lim_n \int_S f_\psi(x_n(s))\phi(s)d\alpha \tag{18}$$

for $\phi \varepsilon L(S)$ and $\psi \varepsilon L(T)$, and it was just seen that $f_\psi(x_0(s))=\lim_n f_\psi(x_n(s))$ a.e. in S for each ψ. Hence

$$f_\psi(U(\phi)) = \int_S f_\psi(x_0(s))\phi(s)d\alpha, \qquad \phi \varepsilon L(S), \psi \varepsilon L(T),$$

or $f_\psi(U(\phi))=f_\psi(\int_S x_0(s)\phi(s)d\alpha)$ since $x_0(.) \varepsilon \mathfrak{A}^\infty(S)[L^\infty(T)]$. The functionals f_ψ forming a total set over $L^\infty(T)$, we arrive at the conclusion that $U(\phi) =\int_S x_0(s)\phi(s)d\alpha$.

D. **Approximation theorems.** The next theorem provides an approximation criterion for weak complete continuity.

THEOREM 3.1.11. *An operation U is defined, separable, and weakly c.c. from $L(S)$ to X if there exists a sequence $x_n(s)$ such that*

(i) $x_n(.) \varepsilon \mathfrak{A}^\infty(S)[X]$, $n=1, 2, \cdots$,

(ii) $\lim_{m,n}$ ess. $\sup_{\cdot s} \|x_m(s)-x_n(s)\|=0$,

(iii) *each $x_n(.)$ is essentially weakly compact-valued,*

(iv) *U coincides with the operation $V(\phi)=\int_S x(s)\phi(s)d\alpha$ defined by the element $x(s)=\lim_n x_n(s)$ of $\mathfrak{A}^\infty(S)[X]$.*

If U is weakly c.c. from $L(S)$ to X, S is Euclidean, and α is Lebesgue measure, then elements $x(.)$, $x_n(.)$, $n=1, 2, \cdots$, exist in $\mathfrak{A}^\infty(S)[X]$ such that conditions (i)–(iv) *are satisfied. Each $x_n(.)$ may be chosen to have only a countable number of functional values.*

If a sequence $\{x_n(.)\}$ exists satisfying (i)–(iv) it is clear that $x(s)$ exists a.e. in S and $x(.)$ is in $\mathfrak{A}^\infty(S)[X]$, so that $V(\phi)=\int_S x(s)\phi(s)d\alpha$ is defined and linear from $L(S)$ to X. Moreover, $U_n(\phi)=\int_S x_n(s)\phi(s)d\alpha$ is for each n also defined and linear from $L(S)$ to X, and by Theorem 3.1.1 each U_n is weakly c.c. From (iii) and (iv) we also have $\lim_n |U-U_n| =\lim_n$ ess. $\sup_{\cdot s} \|x(s)-x_n(s)\| =0$. Thus U is weakly c.c., by virtue of the following fact: *the uniform limit of weakly c.c. operations is also weakly c.c.**

Conversely, if U is weakly c.c. and S is Euclidean, there is by Theorem 3.1.9 an $x(.) \varepsilon \mathfrak{A}^\infty(S)[X]$ such that $U(\phi)=\int_S x(s)\phi(s)d\alpha$ and $x(.)$ is essentially weakly compact-valued. Thus for some $E_0 \varepsilon \mathcal{E}_0$ the set $x(S-E_0)$ is separable and weakly compact in X. From an application of Lindelöf's theo-

* The proof of this statement is almost identical with that of the corresponding theorem for c.c. operations [1, Theorem 2, p. 96].

rem it follows [31, 1.12] that for each n there exists a countably-valued measurable function $x_n(s)$ defined over S, having $x_n(S-E_0) \subset x(S-E_0)$, and satisfying the inequality $\|x(s)-x_n(s)\| < 1/n$ over $S-E_0$. Each $x_n(.)$ is therefore in $\mathfrak{A}^\infty(S)[X]$ and is countably-valued and weakly compact-valued, and $\lim_n$ ess. $\sup._s \|x(s)-x_n(s)\| = 0$.

A similar criterion holds for complete continuity; namely, if either S is Euclidean or X is an $L^q(T)$ space, an operation U is c.c. from $L(S)$ to X if and only if essentially compact-valued $x_n(s)$ exist satisfying (i), (ii), and (iv). Furthermore, the $x_n(s)$ may always be chosen to be simple functions.

Part 2. Applications

In this part we take the opportunity to collect some of the preceding results, to translate them into kernel form, and then to apply them to obtain various conditions sufficient that a given kernel define a c.c. operation from $L(S)$ to X. Usually X will be taken to be an $L^q(T)$ space. A uniform mean ergodic theorem and an application to Markoff processes are discussed in sections D and E.

A. **Criteria for weak compactness in $L(S)$.** In order to apply the preceding results we shall require the following.

Theorem 3.2.1. *Let $[S, \mathcal{E}, \alpha]$ be any system. A subset Φ of $L(S)$ is weakly compact if and only if the following are satisfied:*

(1) *Φ is bounded,*

(2) *given $\epsilon > 0$ there is a $\delta > 0$ such that $\alpha(E) < \delta$ implies $\left|\int_E \phi(s)d\alpha\right| < \epsilon$ for every $\phi \in \Phi$,*

(3) *for each decomposition $\{S_i\}$ of S*

$$\lim_{m\to\infty} \sum_m^\infty \left| \int_{S_i} \phi(s)d\alpha \right| = 0$$

uniformly over Φ.*

If Φ is weakly compact, (1) is obvious. If (2) is false, there exist an $\epsilon > 0$, a sequence $\{\phi_n\}$ in Φ, and measurable sets $\{E_n\}$ such that

(4) $\alpha(E_n) < 1/n$ and $\left|\int_{E_n} \phi_n(s)d\alpha\right| > \epsilon$ *for* $n = 1, 2, \cdots$.

Since Φ is weakly compact, we may suppose $\{\phi_n\}$ to be weakly convergent. In particular, then,

(5) $\lim_{n\to\infty} \int_E \phi_n(s)d\alpha$ exists for every $E \in \mathcal{E}_B$.

But the space $\mathcal{E}_B$ is a complete metric space, and the argument based on the Baire category theorem that was used by Saks [35] for the case $\alpha(S) < \infty$ can be applied here to show that if (5) holds then the integrals $\int_E \phi_n(s)d\alpha$ are ab-

* In case $\alpha(T)$ is finite (3) is implied by (2).

solutely continuous over $\mathcal{E}_B$ uniformly with respect to n. This contradicts (4). Thus (2) holds. To see (3) let $\{S_i\}$ be any decomposition Δ of S. Then $U_\Delta(\phi) = \{\int_{S_i}\phi(s)d\alpha\}$ is an operation sending $L(S)$ into l. Since any operator with range in l takes weakly compact sets into compact sets, $U_\Delta(\Phi)$ is compact in l. Hence [4] (3) holds.

Before proving that (1), (2), and (3) imply that Φ is weakly compact we should like to note that $L(S)$ is weakly complete. For if $\{\phi_i\}$ converges weakly in $L(S)$ then $A_n(E) \equiv \int_E \phi_n(s)d\alpha$ is convergent for each $E \in \mathcal{E}$. Let $A(E) = \lim_n A_n(E)$. If $\{E_i\}$ are disjoint measurable sets then

$$A\left(\sum_{i\in\sigma} E_i\right) = \lim_n A_n\left(\sum_{i\in\sigma} E_i\right) = \lim_n \sum_{i\in\sigma} A_n(E_i)$$

for every set σ of positive integers. Hence if for each n the series $\sum_i A_n(E_i)$ is considered as an element ψ_n of l, it follows [1, p. 137] that $\{\psi_n\}$ is convergent in l. Thus there is a $\psi_0 \in l$ to which $\{\psi_n\}$ converges. The ith term in ψ_0 is the limit of the ith terms of the ψ_n's, that is, the ith term in the series ψ_0 is $\lim_n A_n(E_i) = A(E_i)$. Hence $\sum_i A(E_i)$ is absolutely convergent and $\sum_i A(E_i) = \lim_n \sum_i A_n(E_i) = A(\sum_i E_i)$, which means that $A(E)$ is completely additive over E. Since clearly $A(E) = 0$ when $\alpha(E) = 0$, $A(E)$ is the indefinite integral of some element ϕ_0 of $L(S)$. We thus have $\int_E \phi_0(s)d\alpha = \lim_n \int_E \phi_n(s)d\alpha$ for every E. Since $\{\phi_n\}$ converges weakly and finitely-valued functions are dense in $L^\infty(S)$, it then follows that ϕ_0 is the weak limit of the ϕ_n's.

Now suppose Φ satisfies (1), (2), and (3). Since $L(S)$ is weakly complete, it is enough to show that every sequence in Φ contains a subsequence that converges weakly. The finitely-valued functions being dense in $L^\infty(S)$ and Φ being bounded, it is sufficient to prove that in every sequence in Φ there is a subsequence $\{\phi_i\}$ such that $\lim_i \int_E \phi_i(s)d\alpha$ exists for every $E \in \mathcal{E}$. If $\alpha(S) < \infty$, it is known that (1) and (2) are necessary and sufficient that Φ be weakly compact [11]. Let $\{S_i'\}$ be a fixed decomposition of S. Given any sequence in Φ we can therefore find, since (1) and (2) are satisfied and $\alpha(S_i') < \infty$, a subsequence that is weakly convergent in $L(S_i')$. Thus by diagonalizing we may take the subsequence $\{\phi_n\}$ to have this property:

(6) $\lim_n \int_E \phi_n(s)d\alpha$ exists whenever $E \subset S_i'$ for some i.

Now let E in $\mathcal{E}$ be arbitrary but fixed. Set $S_{2i-1} = E \cdot S_i'$, $S_{2i} = S_i' - E \cdot S_i'$, so that $\{S_j\}$, $j = 1, 2, \cdots$, is a decomposition of S. From (3) there is for each $\epsilon > 0$ an N_ϵ such that

(7) $\sum_{j=2N_\epsilon+1} \left|\int_{S_j} \phi_n(s)d\alpha\right| < \epsilon$, $n = 1, 2, \cdots$.

Each S_j with $j \leqq 2N_\epsilon$ being a subset of some S_i', (6) implies that for some M_ϵ we have

(8) $\left|\int_{E_1} \phi_m(s)d\alpha - \int_{E_1} \phi_n(s)d\alpha\right| < \epsilon$ for $m, n \geqq M_\epsilon$,

where $E_1=\sum_{j=1}^{N_\epsilon}S_{2j-1}$. Writing E_2 for $\sum_{j=N_\epsilon+1}^{\infty}S_{2j-1}$, it follows from (8) and (7) that

$$\left|\int_E \phi_m - \int_E \phi_n\right| \leqq \sum_{i=1}^{2}\left|\int_{E_i}\phi_m - \int_{E_i}\phi_n\right|$$

$$< \epsilon + \sum_{j=N_\epsilon+1}^{\infty}\left|\int_{S_{2j-1}}\phi_m - \int_{S_{2j-1}}\phi_n\right| < 2\epsilon$$

holds for m, $n \geqq M_\epsilon$. Thus $\lim_n \int_E \phi_n(s)d\alpha$ exists for each $E \varepsilon \mathcal{E}$, ending the proof.

THEOREM 3.2.2. *A set Φ in $L(S)$ is weakly compact if* (1) *and* (2) *are true and if*

(3′) $\lim_{m\to\infty}\sum_m^\infty \int_{S_i}|\phi(s)|d\alpha = 0$ *uniformly over Φ for at least one decomposition $\{S_i\}$ of S.*

For (2) and (3′) imply (2) and (3).

B. **Kernel representations of operations among summable functions.** We prove first the following theorem.

THEOREM 3.2.3. *If for a fixed $q<\infty$ the real kernel $K(s, t)$ on $S\times T$ has the properties*

(i) *$K(s, t)$ is measurable on $S\times T$,*

(ii) *for almost all $s \varepsilon S$, $\int_T|K(s, t)|^q d\beta<\infty$,*

(iii) *for every $\psi' \varepsilon L^{q'}(T)$,*

$$\underset{s}{\text{ess. sup.}}\left|\int_T K(s, t)\psi'(t)d\beta\right| < \infty,$$

then

(iv) ess. $\sup._s\,(\int_T|K(s, t)|^q d\beta)^{1/q}\equiv C<\infty$,

(v) *the operation*

$$U(\phi) = \int_S K(s, t)\phi(s)d\alpha$$

is defined and separable from $L(S)$ to $L^q(T)$ and has the further properties

(vi) *U takes weakly compact sets into compact sets,**

(vii) *the norm $|U|$ of U is given by the constant C in* (iv),

(viii) *if $q>1$ then U is weakly c.c.*

For $q=1$ the statement (viii) *is not necessarily true but in this case if the kernel satisfies besides* (i), (ii), *and* (iii) *the condition*

(ix) *there is a null set E_0 such that* (a) $\lim_{\beta(F)=0}\int_F K(s, t)d\beta=0$ *uniformly on*

* See Theorem 3.2.7 and footnote.

$S-E_0$, (b) *for every decomposition* $\{T_i\}$ *of* T

$$\lim_{m\to\infty} \sum_m^\infty \left| \int_{T_i} K(s, t) d\beta \right| = 0 \quad \textit{uniformly on } S - E_0,^*$$

then U *is weakly c.c. on* $L(S)$ *to* $L(T)$. *For* $1 \leqq q < \infty$ *if the kernel* K *satisfies besides* (i), (ii), *and* (iii) *the condition*

(x) *for almost all* $s \in S$ *the set of points* $K(s, .)$ *in* $L^q(T)$ *is compact, then the operation* U *is c.c. from* $L(S)$ *to* $L^q(T)$.

Conversely, if U *is an arbitrary separable operation from* $L(S)$ *to* $L^q(T)$, $1 < q < \infty$, *or if* S *is Euclidean and* α *is Lebesgue measure and* U *is a weakly c.c. operation from* $L(S)$ *to* $L(T)$, *there is a kernel* $K(s, t)$ *representing* U *and satisfying the conditions* (i)–(ix). *If* U *is c.c. from* $L(S)$ *to* $L^q(T)$, $1 \leqq q \leqq \infty$, *there is a kernel* K *representing* U *and satisfying* (i)–(x).

If (i), (ii), and (iii) hold, it is seen from Theorem 1.3.6 that $x(s) = K(s, .)$ is in $\mathfrak{A}^\infty(S)[L^q(T)]$, that $U(\phi) = \int_S x(s)\phi(s) d\alpha = \int_S K(s, t)\phi(s) d\alpha$, and that (iv), (v), and (vii) are true. Since $U(\phi) = \int_S x(s)\phi(s) d\alpha$ the remaining conclusions result from Theorem 3.1.2 and the conditions for weak compactness in $L^q(T)$. The converse part of the theorem follows easily from Theorems 2.2.2, 3.1.7, 3.1.9, 3.1.10, and 1.3.5.

The next theorem, which has been given for Euclidean S by Dunford [9] and Gelfand [18], can be derived from the last theorem above on recalling conditions known to be necessary and sufficient for compactness in $L^q(T)$ when T is a linear interval [33, 40, 41].

THEOREM 3.2.4. *Let* T *be a real interval,* β *be Lebesgue measure, and* $1 \leqq q < \infty$. *If* T *is bounded, then* U *is an operation defined and c.c. from* $L(S)$ *to* $L^q(T)$ *if and only if* $U(\phi) = \int_S K(s, t)\phi(s) d\alpha$ *where* $K(s, t)$ *satisfies* (i), (ii), *and* (iii) *of Theorem* 3.2.3 *and*

$$\lim_{h\to 0} \operatorname{ess. sup.}_s \int_T | K(s, t + h) - K(s, t) |^q d\beta = 0. \tag{3.2.41}$$

If T *is unbounded this statement remains true if* (3.2.41) *is supplemented by the condition that*

$$\lim_{N\to\infty} \operatorname{ess. sup.}_s \int_T | K_N(s, t) - K(s, t) |^q d\beta = 0$$

where $K_N(s, t) = K(s, t)$ *or* 0 *according as* $|t| \leqq N$ *or* $|t| > N$.

* In this part of the theorem, condition (ix)(b) can be replaced by the assumption that $\lim_{m\to\infty} \sum_m^\infty \int_{T_i} |K(s, t)| d\beta = 0$ uniformly over $S - E_0$ for at least one decomposition $\{T_i\}$ of T: see Theorem 3.2.2.

As a second application of Theorem 3.2.3 (or of Theorem 3.2.4) there may be cited

THEOREM 3.2.5. *Let S be a real interval, T be $[0, 1]$, α and β be Lebesgue measure, and $K(r)$, $-\infty < r < \infty$, be periodic with period* 1. *Then if $K(.) \varepsilon L^q(T)$, $1 \leqq q < \infty$, the transformation*

$$U(\phi) = \psi(t) = \int_{-\infty}^{\infty} K(s - t)\phi(s)d\alpha$$

is defined and completely continuous from $L(S)$ to $L^q(T)$ and

$$| U | = \left(\int_0^1 | K(t) |^q d\beta \right)^{1/q}.$$

To prove this it is sufficient in view of Theorem 3.2.3 to verify the conditions (i), (iv), and (x) of that theorem. First we shall show that the kernel $K(s-t)$ is measurable. Using Lusin's theorem let E be a closed set on the interval $0 \leqq s \leqq 1$ such that $\alpha(E) > 1 - \epsilon$ and $K(s)$ is continuous on E. Let E' be the set in $-1 \leqq s \leqq 0$ composed of all points of the form $s-1$ where $s \varepsilon E$. Let D be the set in the square $0 \leqq s, t \leqq 1$ consisting of those (s, t) for which $s - t \varepsilon E + E'$. Then D is a closed set and thus is measurable. To evaluate the measure of D we use Fubini's theorem on the characteristic function $\phi_D(s, t)$ of D. From the above construction it is clear that

$$\int_0^1 \phi_D(s, t)d\alpha = \alpha(E) > 1 - \epsilon, \qquad 0 \leqq t \leqq 1,$$

and so $\alpha \times \beta(D) = \alpha(E) > 1 - \epsilon$. Since $K(s-t)$ is continuous on D, it is, by Lusin's theorem, measurable over $[0, 1] \times [0, 1]$. From periodicity $K(s-t)$ is then measurable over $[n, n+1] \times [0, 1]$ for every integer n, and so (i) is satisfied. Periodicity also implies that

$$\operatorname*{ess.\,sup.}_{s} \int_T | K(s - t) |^q d\beta = \int_T | K(t) |^q d\beta = C < \infty,$$

which proves (iv). To prove (x) we use the known conditions for compactness in $L^q(T)$. Thus $K(s - .)$ is bounded and

$$\lim_{h \to 0} \int_T | K(s - t - h) - K(s - t) |^q d\beta = \lim_{h \to 0} \int_0^{-1} | K(u - h) - K(u) |^q du = 0$$

uniformly in s, so that (x) is satisfied.

The next theorem is analogous to Theorem 3.2.3.

THEOREM 3.2.6. *If the real kernel* $K(s, t)$ *on* $S\times T$ *has the pròperties*

(i) $K(s, t)$ *is measurable on* $S\times T$,

(ii) *there is a null set* E_0 *such that the set* $K(s, .)$, $s \varepsilon S-E_0$, *is a separable subset of* $L^\infty(T)$, *and*

(iii) *for every* $\psi'(.) \varepsilon L(T)$

$$\operatorname{ess.\,sup.}_s \left| \int_T K(s, t)\psi'(t)d\beta \right| < \infty,$$

then

(iv) ess. sup. $|K(s,t)| \equiv C < \infty$,

(v) *the operation* $U(\phi) = \int_S K(s, t)\phi(s)d\alpha$ *is defined and separable from* $L(S)$ *to* $L^\infty(T)$ *and has the further properties*

(vi) U *takes weakly compact sets into compact sets,*

(vii) *the norm* $|U|$ *of* U *is given by the constant* C *in* (iv).

If the kernel $K(s, t)$ *satisfies in addition to* (i), (ii), *and* (iii) *the condition*

(viii) *for a null set* E_0' *the set* $K(s, .)$, $s \varepsilon S-E_0'$, *is weakly compact in* $L^\infty(T)$,

then U *is weakly c.c. on* $L(S)$ *to* $L^\infty(T)$. *If the kernel* $K(s, t)$ *satisfies besides* (i), (ii), *and* (iii) *the condition*

(ix) *for a null set* E_0' *the set* $K(s, .)$, $s \varepsilon S-E_0'$, *is compact in* $L^\infty(T)$,

then the operation U *is c.c. from* $L(S)$ *to* $L^\infty(T)$.

Conversely, if S *is a finite or infinite interval in Euclidean space and* α *is Lebesgue measure, and* U *is a weakly c.c. operator from* $L(S)$ *to* $L^\infty(T)$, *then there is a kernel* $K(s, t)$ *representing* U *and satisfying the conditions* (i)–(viii). *If* S *is abstract and* U *is c.c., then* (ix) *is also satisfied.*

Properties (iv), (v), and (vii) come from Theorem 1.3.7. The remaining conclusions in the first part result from Theorem 3.1.2. The converse is derived by means of Theorems 3.1.9, 3.1.10, and 1.3.5.

The next result is essentially based on Theorems 3.1.2 and 3.2.1.

THEOREM 3.2.7. *Suppose* $U(\phi) = \int_S K(s, t)\phi(s)d\alpha$ *is an operation defined and separable from* $L(S)$ *to* $L^q(T)$ *and suppose* $\Phi \subset L(S)$ *satisfies conditions* (1)–(3) *of Theorem* 3.2.1. *Then the set*

$$\Psi = \left[\int_S K(s, t)\phi(s)d\alpha, \ \phi \varepsilon \Phi \right]$$

is compact in* $L^q(T)$ *when* $1 < q < \infty$. *This is also true when* $q = 1$ *provided that* K *is measurable and either* ess. sup.$_s \int_T |K(s, t)| d\beta < \infty$ *or* S *is Euclidean and* T *is linear.*

* Thus for example if $\alpha(E') < \infty$ and E varies over $\mathcal{E}(E')$ (see footnote to Theorem 2.1.0), the functions $\psi_E(t) = \int_E K(s, t)d\alpha$ form a compact set in $L^q(T)$.

Here in all three cases (Theorem 2.2.1, Theorem 2.4.5, and a remark at the end of the latter) U has a representation $\int_S x(s)\phi(s)d\alpha$ where $x(.) \in \mathfrak{A}^\infty(S)[L^q(T)]$. Hence U takes weakly compact sets into compact sets.

Another corollary to Theorem 3.2.3 is

THEOREM 3.2.8. *Suppose X is arbitrary. If $x(.) \in \mathfrak{A}^\infty(T)[X]$ and if $K(s, t)$ satisfies conditions* (i)–(iii) *and* (ix) *of Theorem* 3.2.3 *for $q=1$, then $U(\phi) = \int_T x(t)\{\int_S K(s, t)\phi(s)d\alpha\}d\beta$ is c.c. from $L(S)$ to X.*

For $\int_S K(s, t)\phi(s)d\alpha$ is weakly c.c. from $L(S)$ to $L(T)$ and $\int_T x(t)\psi(t)d\beta$ takes weakly compact sets in $L(T)$ into compact sets in X.

Of the several corollaries to the above theorem we mention only the following, obtained by letting $X = L^q(W)$.

THEOREM 3.2.9. *Suppose that $K(s, t)$ is given as in the preceding theorem. Let $[W, \mathcal{G}, \gamma]$ be a third system analogous to $[S, \mathcal{E}, \alpha]$ and $[T, \mathcal{J}, \beta]$. If $M(t, w)$ is measurable over $T \times W$, $M(t, .) \in L^q(W)$ for almost all t, and* ess. $\sup._t |\int_W M(t, w)\chi(w)d\gamma| < \infty$ *for every $\chi(.) \in L^{q'}(W)$, then*

$$U(\phi) = \int_S \phi(s)\left\{\int_T K(s, t)M(t, w)d\beta\right\}d\alpha$$

is an operation defined and c.c. from $L(S)$ to $L^q(W)$, $q < \infty$. When $q = \infty$ this is still true provided that $M(t, .)$ is almost separably-valued in $L^q(W)$.

This can be shown to be a consequence of Theorem 3.2.8 and the Fubini theorem. A more direct way perhaps is as follows. We know that

$$L^q(W) \ni \int_T M(t, w)\psi(t)d\beta \qquad \text{for } \psi \in L(T),$$

and that in $\mathcal{E}_0$ there is an E_0 such that $K(s, .)$ forms a weakly compact set in $L(T)$ as s varies over $S - E_0$. Hence for each s in $S - E_0$

$$L^q(W) \ni \int_T M(t, w)K(s, t)d\beta \equiv H(s, w),$$

and $H(s, .)$, $s \in S - E_0$, forms a compact set in $L^q(W)$ since by Theorem 3.2.3 or 3.2.6 the operation $\int_T M(t, w)\psi(t)d\beta$ sends weakly compact sets into compact sets. Let $y(s) = H(s, .)$. From the measurability of M and K it follows that $H(s, w)$ is measurable, so that by Theorem 1.3.5 the vector function $y(.)$ is measurable. Since $y(.)$ is measurable and essentially compact-valued, the following operation is c.c. from $L(S)$ to $L^q(W)$:

$$\int_S y(s)\phi(s)d\alpha = \int_S H(s, w)\phi(s)d\alpha = \int_S \phi(s)\left\{\int_T K(s, t)M(t, w)d\beta\right\}d\alpha.$$

It is easily seen that a particular case of the last theorem is the following result of Servint [37; also see Theorems 1 and 3 of 46]: if $S=T=W=[0, 1]$ and all measures are that of Lebesgue, and if $K(s, t)$ and $M(t, w)$ are each bounded and measurable, then the above operation U is defined and c.c. from $L(S)$ to $L(W)$. In this case we can, by the above theorem, go even further and assert that U is defined and c.c. from $L(S)$ to $L^q(W)$ for each $q<\infty$, and that this is still true when $q=\infty$ provided that $M(t, .)$ is almost separably-valued in $L^\infty(W)$.

Translation of other theorems into kernel terminology will be left to the reader.

C. **C.c. operations from $L^p(S)$ to X.** We append the following remarks concerning c.c. operations from $L^p(S)$ to X. Here S is abstract and X is an arbitrary B-space.

THEOREM 3.2.10. *Let $x(.)$ on S to X be measurable and suppose that for some $p'<\infty$*

$$\|x(.)\| \equiv \left(\int_S \|x(s)\|^{p'} d\alpha\right)^{1/p} < \infty .$$

Then the equation

$$U(\phi) = \int_S x(s)\phi(s)d\alpha$$

defines a completely continuous operator from $L^p(S)$ to X with $|U| \leqq \|x(.)\|$.

The statement involving the norm follows from the Hölder inequality [9]. By the definition of measurability there is a sequence $x_n(s)$ of simple functions approaching $x(s)$ a.e. in S. Let $y_n(s)$ be defined to coincide with $x_n(s)$ on the set where $\|x_n(s)-x(s)\| \leqq \|x(s)\|$ and to vanish elsewhere. By the Lebesgue convergence theorem then

$$\lim_n \int_S \|x(s) - y_n(s)\|^{p'} d\alpha = 0.$$

Since the y_n's are simple they obviously define c.c. operators U_n which, by the above equation, satisfy the condition $\lim_n |U-U_n| =0$. This shows that U is c.c.

THEOREM 3.2.11. *Let $K(s, t)$ be measurable on $S\times T$ and suppose that for some $p'<\infty$ and $q<\infty$*

$$\left(\int_S \left(\int_T | K(s, t) |^q d\beta\right)^{p'/q} d\alpha\right)^{1/p'} \equiv C < \infty .$$

Then the equation

$$\psi(t) = \int_S K(s, t)\phi(s)d\alpha$$

defines a completely continuous operator from $L^p(S)$ *to* $L^q(T)$.

This is an immediate corollary of the preceding theorem together with Theorems 1.3.6 and 1.3.5. This theorem was first proved for Euclidean S and T and $p=q$ by E. Hille and J. D. Tamarkin [21]. Another proof for Euclidean S and T was published later by F. Smithies [38].

As another application of Theorem 3.2.10 we mention the following.

THEOREM 3.2.12. *Let* $K(s, t)$ *be measurable on* $S \times T$ *and such that the function* $K(s, .)$ *is almost separably-valued in* $L^\infty(T)$. *Then if* $p' < \infty$ *and*

$$\int_S \operatorname{ess.\,sup.}_t \, |K(s, t)|^{p'} d\alpha < \infty,$$

the operation

$$U(\phi) = \int_S K(s, t)\phi(s)d\alpha$$

is defined and c.c. from $L^p(S)$ *to* $L^\infty(T)$.

This is immediately derivable from Theorems 1.3.6, 1.3.5, and 3.2.10.

Further applications of Theorem 3.2.10 will be left to the reader.

D. **A mean ergodic theorem.** Let U be an operation sending $L(S)$ into $L(S)$ and having a representation $U(\phi) = \int_S x(s)\phi(s)d\alpha$ where $x(.)$ is an essentially weakly compact-valued element of $\mathfrak{A}^\infty(S)[L(S)]$. Since U is separable, $U(L(S))$ has a separable span X in $L(S)$. From the footnote to Theorem 2.2.6 there is a Borel field $\mathcal{J}$ contained in $\mathcal{E}$ such that $\mathcal{J}_B$ is separable and X is a subset of those elements of $L(S)$ which are $\mathcal{J}$-measurable. If $\mathcal{J}$ is any such Borel field and if we set $T=S$ and $\beta=\alpha$, the triad $[T, \mathcal{J}, \beta]$ is a system for which $L(T)$ is a separable c.l.m. within the $\mathcal{J}$-measurable elements of $L(S)$, and $L(T) \supset X \supset U(L(S))$.

The particular mean ergodic theorem we wish to give is one dealing with operations of this sort.* Basically it is a special case of an extension made to general B-spaces by Fortet [14, 13], Yosida [43, 44], and Kakutani [22] of a result due to Kryloff and Bogoliouboff [26] and Fréchet [15].

* In *Dependent probabilities and spaces* (L), Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 154–159, Garrett Birkhoff has discussed for generalized $L(S)$ spaces the discrete mean ergodic theory of what he has termed "transition operators" (also see S. Kakutani, *Mean ergodic theorem in abstract* (L)*-spaces*, Proceedings of the Imperial Academy, Tokyo, vol. 15 (1939), pp. 121–123). These are norm-preserving operators that send non-negative elements into non-negative elements. Some account of the continuous theory in an $L(S)$ space can be found in [11].

THEOREM 3.2.13. *If U is an operation of the above type, these conclusions can be made*: (I) *there is a kernel $K_1(s, t)$ defined over $S\times T=S\times S$ such that*

(i) *$K_1(s, t)$ is measurable*,

(ii) *$K_1(s, .)=x(s)$ in $L(T)$ for almost every s, and*

(iii) *$U(\phi)=\int_S K_1(s, t)\phi(s)d\alpha$ for each $\phi \in L(S)$*;

(II) (i) *U is separable and weakly c.c.*,

(ii) *U takes weakly compact sets into compact sets*,

(iii) *U^n is c.c. for $n\geqq 2$*,

(iv) $|U| =$ ess. sup. $\|x(s)\| =$ ess. sup.$_s \int_T |K_1(s, t)| d\beta$,

(v) *the maximum number of linearly independent solutions of the equation $\phi=U(\phi)$ is a finite number N.*

Suppose in addition that the norms $|U^n|$ are bounded. Then

(vi) $\lim_{m,n}|V_m-V_n|=0$ *where* $V_m=(1/m)\sum_1^m U^i$.

(III) *The limit operation $V=\lim V_m$ has these properties*:

(i) *V is c.c.*,

(ii) $VU^n=U^nV=V$ *for* $n=1, 2, \cdots$,

(iii) $V^2=V$,

(iv) *the fixed points of V form a c.l.m. Y having dimension N*,

(v) *Y is exactly the set of fixed points of U*,

(vi) $V(L(S))=Y$.

(IV) *If $x_0(s)\equiv V(x(s))$ then*

(i) $x_0(.) \in \mathfrak{A}^\infty(S)[Y]$,

(ii) $V(\phi)=\int_S x_0(s)\phi(s)d\alpha$,

(iii) $|V| =$ ess. sup. $\|x_0(s)\|$,

(iv) $x_0(s)=V(x(s))=U^n(x_0(s))=V(U^n(x(s)))=V(x_0(s))$ *for every n and almost all s*,

(v) *$x_0(s)$ can be written as $x_0(s)=\sum_1^N\mu_i(s)\phi_i$ where*

(vi) *each ϕ_i is in Y and $\|\phi_i\|=1$*,

(vii) *the set $\phi_1, \cdots, \phi_N$ forms a basis for Y*,

(viii) *$\mu_i(.) \in L^\infty(S)$ and $\int_S\mu_i(s)\phi_i(s)=1$ or equals 0 according as $i=j$ or $i\neq j$, and*

(ix) *in $L^\infty(S)$ the c.l.m. Y^* generated by the set $\mu_i(.)$, $i=1, 2, \cdots, N$, is both the set of fixed points of U^* (the operation adjoint to U) and the set of fixed points for the c.c. adjoint V^* of V, and Y^* has dimension N.*

(V) *Finally, the following statements are true concerning the measurable kernels $K_0(s, t)\equiv\sum_1^N\mu_i(s)\phi_i(t)$, $K_1(s, t)$, and $K_n(s, t)=\int_S K_{n-1}(s, s')K_1(s', t)d\alpha$, $n=2, 3, \cdots$*:

(i) $V(\phi)=\int_S K_0(s, t)\phi(s)d\alpha$,

(ii) $U^n(\phi)=\int_S K_n(s, t)\phi(s)\, d\alpha$ *for* $n=1, 2, \cdots$,

(iii) *for almost every s the equality*

$$K_0(s, t) = \int_S K_{n+1}(s, s')K_0(s', t)d\alpha = \int_S K_0(s, s')K_n(s', t)d\alpha$$

holds for $n=0, 1, 2, \cdots$ *and for almost all* t;

(iv) $K_n(s, .)$ *is essentially weakly compact-valued in* $L(T)$ *for* $n=1$ *and essentially compact-valued for* $n\neq 1$.

There is a null set E_0 *such that*

(v) $\lim_{n\to\infty} \int_T |K_0(s, t) - (1/n)\sum_1^n K_i(s, t)| d\beta = 0$ *uniformly over* $S-E_0$,

(vi) $\lim_{n\to\infty}(1/n)\sum_1^n \int_F K_i(s, t)d\beta = \int_F K_0(s, t)d\beta$ *uniformly over* $(S-E_0)\times F$,

(vii) $\lim_{n\to\infty} \int_T |\int_S \phi(s)K_0(s, t)d\alpha - \int_S(\phi(s)/n)\sum_1^n K_i(s, t)d\alpha| d\beta = 0$ *uniformly over those* ϕ *in* $L(S)$ *having* $\int_S |\phi(s)| d\alpha \leqq 1$.

Statements (I)(i)–(iii) and (II)(iv) are contained in Theorem 1.3.5, and Theorem 3.1.6 yields (II)(i)–(iii) and (II)(v). If the norms $|U^n|$ are bounded, conclusions (II)(vi) and (III)(i)–(iii) follow from [43] or Theorem 5 of [22] and the fact that U^2 is c.c. The definition of V taken together with (III)(ii) shows that the fixed points of V, which form a c.l.m. Y, are also the fixed points of U; this vindicates (III)(iv)–(v). Conclusion (III)(vi) comes from (III)(iii) and the definition of Y. Letting $x_0(s) \equiv V(x(s))$, (III)(vi) shows that $x_0(S) \subset Y$; and since measurability and being essentially bounded are two properties that are operational invariants, (IV)(i) is true. To verify (IV)(ii) it is sufficient to note that $V(\phi) = V(U(\phi)) = V(\int_S x(s)\phi(s)d\alpha) = \int_S V(x(s))\phi(s)d\alpha$; (IV)(iii) is then obvious. Conclusion (IV)(iv) results easily from (III)(ii)–(iii), the definition of $x_0(s)$, and the assumption that $x(s)$ is defined for almost every s.

To establish the rest of (IV) we may proceed as follows. Since Y is a c.l.m. in $L(S)$ having finite dimension N, it is possible to choose in Y a basis $\phi_1, \cdots, \phi_N$ satisfying (IV)(vi). Choosing such a basis we have

$$\phi = \sum_1^N g_i(\phi)\phi_i, \qquad \phi \in Y,$$

where each g_i is a linear functional over Y and $g_i(\phi_j) = \delta_{ij}$, the Kronecker delta. Thus

$$x_0(s) = \sum_1^N g_i(x_0(s))\phi_i = \sum_1^N \mu_i(s)\phi_i$$

where $\mu_i(.) \equiv g_i(x_0(.))$ is in $L^\infty(S)$ for each i since $x_0(.) \in \mathfrak{A}^\infty(S)[Y]$. Statements (IV)(v)–(viii) are therefore verified. To establish (IV)(ix) we first note that since $\lim_m |V - V_m| = 0$ we also have $\lim_m |V^* - V_m^*| = 0$ where V^* and V_m^* are the operations adjoint to V and V_m respectively and $V_m^* = (1/m)\sum_1^m (U^i)^* = (1/m)\sum_1^m (U^*)^i$. It then follows that U^* and V^* have the same set of fixed points, and this set is a c.l.m. in $L^\infty(S)$ that has dimension N since V^* is the adjoint of the c.c. operation V. This c.l.m. is Y^*. For (IV)(ii) and Theorem 1.1.3 imply that for any f in the adjoint of $L(T)$ the linear functional $V^*(f)$ over $L(S)$ is defined by the element $f(x_0(.))$ of $L^\infty(S)$. Since $f(x_0(s))=\sum_1^N \mu_i(s)f(\phi_i)$, $V^*(f)$ is defined by an element of Y^*. Thus $V^*(L^\infty(T)) \subset Y^*$, and hence $f \varepsilon Y^*$ when $V^*(f)=f$. Conversely, if $f \varepsilon Y^*$ then $V^*(f)=f$. It is sufficient to consider the functional f_j defined by the element $\mu_j(.)$ of $L^\infty(T)$. For each ϕ_i we have

$$f_j(\phi_i) = \int_S \mu_j(s)\phi_i(s)d\alpha = \int_S g_j(x_0(s))\phi_i(s)d\alpha = g_j(V(\phi_i)) = g_j(\phi_i),$$

so that $f_j(x_0(s))=\sum_1^N f_j(\phi_i)\mu_i(s)=\sum_1^N g_j(\phi_i)\mu_i(s)=\mu_j(s)$. Thus $V^*(f_j)$ is defined by $\mu_j(s)$, that is, $V^*(f_j)=f_j$. This completes the proof of (IV)(ix).

In (V) the first conclusion results from (IV)(ii) and (IV)(v). Setting

(1) $x_1(s)\equiv x(s)$ and $x_n(s)=U(x_{n-1}(s))$ for $n\geqq 2$,

it is then clear by induction that the three statements

(2) $x_n(.) \varepsilon \mathfrak{A}^\infty(S)[L(S)]$,

(3) $U^n(\phi)=\int_S x_n(s)\phi(s)d\alpha$,

(4) $K_n(s, .)=x_n(s)$ in $L(S)$ for almost all s

must each hold for $n=1, 2, \cdots$. Since each $K_n(s, t)$ is measurable, (V)(ii) then follows from (2), (3), (4), and Theorem 1.3.5. Conclusion (V)(iii) comes from (1), (3), (4), and (V)(ii), and (V)(iv) is a corollary to (2), (3), (4), (II)(i), (II)(iii), and Theorem 3.1.2. Finally, (V)(v) and (V)(vii) are true since

$$\begin{aligned} 0 = \lim_m \left| V - V_m \right| &= \lim_m \text{ ess. sup.} \left\| x_0(s) - \frac{1}{m}\sum_1^m x_i(s) \right\| \\ &= \lim_m \operatorname*{ess.\,sup.}_s \int_T \left| K_0(s, t) - \frac{1}{m}\sum_1^m K_i(s, t) \right| d\beta, \end{aligned}$$

and (V)(vi) is obvious from (V)(v).

This and Theorem 3.1.9 lead at once to

THEOREM 3.2.14. *If S is a finite or infinite Euclidean interval and α is Lebesgue measure and U is an arbitrary weakly c.c. operation in $L(S)$, then $U(\phi)=\int_S x(s)\phi(s)d\alpha$ where $x(.)$ is an essentially weakly compact-valued element of $\mathfrak{A}^\infty(S)[L(S)]$. Hence conclusions* (I) *and* (II)(i)–(v) *of Theorem* 3.2.13 *hold in this case, and if the norms $|U^n|$ are bounded the remaining conclusions are also valid for U.*

A simple example shows that in Theorem 3.2.13 we cannot drop entirely the assumption that $x(.)$ is essentially weakly compact-valued. On the real axis let J_n be the interval $[n, n+1)$ for each integer n and on each J_n assign $x(s)$ the constant value ϕ_{n+1} where ϕ_{n+1} is the characteristic function of J_{n+1}. Obviously $x(.) \in \mathfrak{A}^\infty(S)[L(S)]$ and ess. sup. $\|x(s)\| = 1$, so that $U(\phi) = \int_S x(s)\phi(s)ds$ satisfies all the hypotheses of Theorem 3.2.13 except that $x(.)$ is not essentially weakly compact-valued. Since $U(\phi_n) = \phi_{n+1}$, it follows that the principal conclusions of the theorem are false in this case. For $\Phi = \{\phi_n\}$ is bounded in $L(S)$ but not weakly compact, and thus U^n fails for each n to be weakly c.c. since $U(\Phi) = \Phi$. Moreover, $\|V_m(\phi_0) - V_{2m}(\phi_0)\| = 1$ for every m, as the reader can easily verify; hence the operations V_m do not converge even pointwise.

E. **An application to Markoff processes.** Let $\mathcal{J}$ be a second Borel field of subsets of S, with $\mathcal{J} \subset \mathcal{E}$ and $\mathcal{J}_B$ separable. Take $T = S$ and $\beta = \alpha$, so that $[T, \mathcal{J}, \beta]$ forms a second system for which $L(T)$ is a separable c.l.m. in $L(S)$. Suppose further that there is a function $P(s, F)$ defined over $S \times \mathcal{J}$ and having these properties:

(1) $P(s, F)$ is $\mathcal{E}$-measurable in s for each $F \in \mathcal{J}_B$,

(2) $s \in S$ implies that $P(s, F)$ is non-negative and completely additive over $\mathcal{J}$ and $P(s, T) = 1$,

(3) if $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ and $\beta(\prod_1^\infty F_n) = 0$, then $\lim_n P(s, F_n) = 0$ uniformly over S.

As Doob has shown [6], under these circumstances $P(s, F)$ is a set of conditional probability functions for a certain class of Markoff processes. Functions $P(s, F)$ of this sort have been discussed frequently in probability (for references see [6]), and most of the statements in the next theorem are already known, being due either to Fréchet [15], Kryloff and Bogoliouboff [26], Doeblin and Fortet [5], Doob [6], Yosida and Kakutani [45], or to earlier authors. However, the indigeneity of the theorem and the fact that some of the conclusions are apparently new lead us to include it here.

From (2), (3), and the Radon-Nikodym theorem there is for each s a point $x(s)$ in $L(T)$ whose indefinite integral over F is precisely $P(s, F)$. We now make the assertion that

(4) *$x(.)$ is a weakly compact-valued element of $\mathfrak{A}^\infty(S)[L(T)]$, $\|x(s)\| \equiv 1$, and each point $x(s)$ is an essentially non-negative element of $L(T)$.*

In the first place it is evident from (2) that for each s the element $x(s)$ in $L(T)$ is essentially non-negative and has $\|x(s)\| = 1$. Secondly, in view of (1), the decomposability of T, and the complete additivity in F of $P(s, F)$, it results that $P(s, F)$ is measurable in s for each $F \in \mathcal{J}$. Hence $f(x(s))$ is measurable for each linear functional f defined over $L(T)$ by a finitely-valued element of $L^\infty(T)$. Since such elements are dense in $L^\infty(T)$, $f(x(s))$ must be measurable for every linear functional over $L(T)$. The space $L(T)$ being separable, it then follows from Theorem 1.1.7 that $x(.)$ is measurable. Thus $x(.) \varepsilon \mathfrak{A}^\infty(S)[L(T)]$ and $\|x(s)\| \equiv 1$. To see that $x(.)$ is weakly compact-valued we have only to note that property (3) of $P(s, F)$ implies that $\Phi = x(S)$ satisfies conditions (2) and (3) of Theorem 3.2.1.

Since $L(T)$ forms a c.l.m. in $L(S)$ and $L(T) \supset x(S)$, we can also state that

(5) *assertion* (4) *above holds with* $L(T)$ *replaced by* $L(S)$.

We are now ready to give the theorem. Here, as before, $\mathfrak{P}$ denotes the essentially non-negative elements of $L(S)$ and $\mathfrak{P}_1 = \mathfrak{P}[\|\phi\| = 1]$.

THEOREM 3.2.15. *Let U be the operation $U(\phi) = \int_S x(s)\phi(s)d\alpha$ defined by the function $x(.)$ of* (4). *Then all the conclusions of Theorem* 3.2.13 *hold here and in addition*

(I) (iv) $\int_F K_1(s, t)d\beta = P(s, F)$ *for every* $s \varepsilon S$ *and every* $F \varepsilon \mathcal{J}$,

(v) $K_1(s, t) \geqq 0$,

(vi) $\int_T K_1(s, t)d\beta = 1$ *for every* s,

(II) (vii) $|U| = 1$,

(viii) *if* $\phi \varepsilon \mathfrak{P}$ *then* $U(\phi) \varepsilon \mathfrak{P}$ *and* $\|U(\phi)\| = \|\phi\|$,

(III) (vii) $|V| = 1$,

(viii) *if* $\phi \varepsilon \mathfrak{P}$ *then* $V(\phi) \varepsilon \mathfrak{P}$ *and* $\|V(\phi)\| = \|\phi\|$,

(ix) *the dimension number N of Y is positive*,

(IV) (x) *each* ϕ_i *in the basis* $\phi_1, \cdots, \phi_N$ *may be chosen in* $\mathfrak{P}_1$,

(V) (viii) *for* $n = 0, 1, 2, \cdots, K_n(s, t) \geqq 0$ *a.e. in* $S \times T$ *and* $\int_T K_n(s, t)d\beta = 1$ *a.e. in* S.

From (5) and the equality $|U| =$ ess. sup. $\|x(s)\| = 1$ it is evident that U satisfies the requirements of Theorem 3.2.13, including the one that the norms $|U^n|$ be bounded. Moreover, the span of $U(L(S))$ must by Theorem 1.2.10 be in the span X belonging to $x(S)$; thus $U(L(S)) \subset X \subset L(T)$, and X is clearly a subset of those elements of $L(S)$ which are $\mathcal{J}$-measurable. Hence all the conclusions of Theorem 3.2.13 hold in the present situation.

The kernel K_1 given in Theorem 3.2.13 can fail to satisfy (I)(iv) and (I)(v) on at most a set of $S \times T$ measure zero. We may suppose K_1 altered on this set so as to satisfy these two conditions; the previous properties of K_1 and of the kernels K_n will not be affected since the exceptional set was of measure 0. Conclusion (II)(vii) was shown above, and (I)(vi) is obvious from (I)(iv). According to (4) the closed convex set $\mathfrak{P}_1$ contains $x(S)$; from (II) of Theorem 1.2.10 we conclude that

(6) $U(\phi) \varepsilon \mathfrak{P}_1$ for each $\phi \varepsilon \mathfrak{P}_1$.

This yields (II)(viii). On writing $x_1(s) \equiv x(s)$ and $x_i(s) = U(x_{i-1}(s))$ for $i \geqq 2$ we noted, in proving Theorem 3.2.13, that

(7) $U^i(\phi) = \int_S x_i(s)\phi(s)d\alpha$.

From (6) it follows that $x_i(s) \in \mathfrak{P}_1$ for each $i \geqq 1$ and each s, so that $(1/n)\sum_1^n x_i(s)$ is in the convex set $\mathfrak{P}_1$ for every n and s. Now (7) implies that

$$\lim_n \text{ ess. sup.}_s \left\| x_0(s) - \frac{1}{n} \sum_1^n x_i(s) \right\| = \lim_n | V - V_n | = 0,$$

and hence $x_0(s) = \lim_n (1/n)\sum_1^n x_i(s)$ for almost every s. Since $\mathfrak{P}_1$ is closed we must have $x_0(s) \in \mathfrak{P}_1$ for almost all s. This combined with Theorem 1.2.10 shows that (III)(viii) holds and that $|V| = \text{ess. sup. } \|x_0(s)\| = 1$. Conclusion (III)(ix) now follows from the fact that $Y = V(L(S))$ and that $|V| \neq 0$. To prove (IV)(x) let $\phi_1', \cdots, \phi_N'$ be any basis for Y with $\|\phi_i'\| = 1$. From (III)(iv), (vi), and (vii) the unit sphere Y_1 of Y is the image under V of the unit sphere of $L(S)$. Theorem 1.2.10 then implies that for each null set E_0 the set Y_1 is in the closed convex hull of the point set sum $Y_2 = x_0(S - E_0) + R(x_0(S - E_0))$. Choosing E_0 such that $x_0(s) \in \mathfrak{P}_1$ for $s \notin E_0$, it follows that the closed convex hull of $x_0(S - E_0)$ is the convex hull of $x_0(S - E_0)$. The same is therefore true of Y_2. Hence each $\phi \in Y_1$ can be written as $\phi = \sum_{j=1}^k c_j \bar{\phi}_j$ where each $\bar{\phi}_j \in \mathfrak{P}_1$ and $\sum_1^k |c_j| = 1$. On expressing each ϕ_i' as $\phi_i' = \sum_{j=1}^{k_i} c_{ij}\bar{\phi}_{ij}$ it is clear that each element of Y is a linear combination of the $\bar{\phi}_{ij}$'s. Hence from among these a subset $\phi_1, \cdots, \phi_N$ can be chosen to form a basis for Y. This establishes (IV)(x). Finally, (V)(viii) results on recalling that $x_n(s) = K_n(s, .)$ in $L(T)$ for almost all s and that $x_n(s) \in \mathfrak{P}_1$ for every s.

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